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Abstract: In the Brownian case, the links between dynamic risk measures and BSDEs have been widely studied. In this paper, we study the case with jumps. We first study the properties of BSDEs driven by a Brownian motion and a Poisson random measure. In particular, we provide a comparison theorem, under quite weak assumptions, extending that of Royer [25]. We then give some properties of dynamic risk measures induced by BSDEs with jumps. We provide a representation property of such dynamic risk measures in the convex case as well as some new results on a robust optimization problem, related to the case of model ambiguity.

Key-words: Backward stochastic differential equations with jumps, comparison theorems, risk measures

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CENTRE DE RECHERCHE

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Equations Différentielles Stochastiques Rétrogrades avec sauts, optimisation et applications aux mesures de risque dynamiques

Résumé : Les liens entre les EDSR et les mesures de risque dynamiques ont été largement étudiées dans le cas Brownien. On étudie dans ce papier le cas avec sauts. On étudie tout d'abord les propriétés des EDSR dirigées par un mouvement Brownien et une mesure de Poisson aléatoire : on prouve en particulier un théorème de comparaison sous des hypothèses assez faibles qui généralise celui de Royer 2006. On donne ensuite des propriétés pour les mesures de risques dynamiques induites par les EDSR avec sauts. On établit un théorème de représentation duale pour de telles mesures de risques dans le cas convexe. On étudie enfin un problème d'optimisation robuste de mesure de risques associé au cas d'ambiguité de modèle.

Mots-clés : Equations Différentielles Stochastiques Rétrogrades avec sauts, mesures de risques, théorème de comparaison

1 Introduction

Linear backward stochastic differential equations (BSDE) were introduced by Bismut (1976) [5] as the adjoint equations associated to stochastic Pontryagyn maximum principles in stochastic control theory. The general case of nonlinear BSDEs was then studied by Pardoux and Peng (1990) (see [18] and [19]) in the Brownian framework. In [19], they provided Feynman Kac representations of solutions of non-linear parabolic partial differential equations. In the paper of El Karoui et al. (1997) [10], some additional properties are given and several applications to option pricing and recursive utilities are studied.

The case of a discontinuous framework is more involved, especially concerning the comparison theorem, which requires an additional assumption. In 1994, Tang and Li [26] provided an existence and uniqueness result in the case of a natural filtration associated with a Brownian motion and a Poisson random measure. In 1995, Barles, Buckdahn, Pardoux [3] provided a comparison theorem as well as some links between BSDEs and non-linear parabolic integral-partial differential equations, generalizing some results of [18] to the case of jumps. In 2006, Royer [25] proved a comparison theorem under weaker assumptions, and introduced the notion of non linear expectations in this framework.

Furthermore, in 2004-2005, various authors have introduced dynamic risk measures in a Brownian framework, defined as the solutions of BSDEs (see [20, 4, 14, 23]). More precisely, given a Lipschitz driver $f(t, x, \pi)$ and a terminal time T, the risk measure ρ at time t of a position ξ is given by $-X_t$, where X is the solution of the BSDE driven by a Brownian motion, associated with f and terminal condition ξ. By the comparison theorem, ρ satisfies the *monotonicity property*, which is usually required for a risk measure (see [13]). Many studies have been recently done on such dynamic risk measures, especially concerning relative robust optimization problems and optimal stopping problems (see, among others, Bayraktar and coauthors in [2]).

In this paper, we are concerned with dynamic risk measures induced by BSDEs with jumps. We study their properties as well as some related optimization problems.

We begin by studying BSDEs with jumps and their properties. We first focus on linear BSDEs which play an important role in the comparison theorems as well as in the applications to finance. We show that the solution is given by a conditional expectation via an exponential semimartingale, usually called the adjoint process. We also provide some additional properties of the solution and its adjoint process, which are specific to the jump case.

Using these properties, we provide a comparison theorem as well as a strict comparison theorem, under mild assumptions, which generalise those stated in [25]. We also prove some optimization principles for BSDEs with jumps. More precisely, we consider a family of controlled drivers f^α, $\alpha \in \mathcal{A}$ and show that, under some hypothesis, the infimum of the associated solutions X^α can be characterized as the solution of a BSDE. Moreover, the driver of this BSDE is given by the infimum of the drivers f^α, $\alpha \in \mathcal{A}$. We provide a sufficient condition of optimality. Also, from the strict comparison theorem, we derive a necessary condition of optimality.

We then give some properties of dynamic risk measures induced by BSDEs with jumps. Note that contrary to the Brownian case, the monotonicity property does not generally holds, and requires an additional assumption.

In the case of a concave driver f, we provide a dual representation property of the associated convex risk measure via a set of probability measures which are absolutely continuous with respect to the initial probability P.

At last, we study the case of ambiguity on the model: More precisely, we consider a model parameterized by a control α as follows. To each coefficient α, is associated a probability measure Q^α, equivalent to P, called *prior,* as well as a monotone risk measure ρ^α induced, under Q^α, by a BSDE with jumps. We consider an agent who is averse to ambiguity and define her risk measure as the supremum over α of the risk measures ρ^α. We show that this dynamic risk measure is induced, under P, by a BSDE.

The paper is organized as follows. In Section 2, we introduce the notation and the basic definitions. Section 3 is dedicated to linear BSDEs with jumps. In Section 4, comparison theorems for BSDEs with jumps are provided. We also prove an optimization principle which allows us to characterize the value function of an optimization problem written in terms of BSDEs. Section 5 is dedicated to dynamic risk measures induced by BSDE with jumps and related robust optimization problems. In Subsection 5.1, we give properties of dynamic risk measures induced by BSDEs with jumps. In the case of a concave driver, we provide a dual representation of the associated convex risk measure (Subsection 5.2). The problem of dynamic risk measures under model ambiguity is addressed in Subsection 5.3. Finally, in Section 5.4, we interpret the dependence of the driver with respect to x in terms of the instantaneous interest rate. In the Appendix, we provide some useful additional properties on exponential local martingales, and BSDEs with jumps.

2 BSDEs with jumps: notation and definition

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let W be a one-dimensional Brownian motion and $N(dt, du)$ be a Poisson random measure with compensator $\nu(du)dt$ such that ν is a σ-finite measure on $\mathbf{R}^*$, equipped with its Borel field $\mathcal{B}(\mathbf{R}^*)$. Let $\tilde{N}(dt, du)$ be its compensated process. Let $I\!F = \{\mathcal{F}_t, t \geq 0\}$ be the natural filtration associated with W and N. The results of this paper can be generalized to multi-dimensional Brownian motions and Poisson random measures without difficulty.

Notation. Let $\mathcal{P}$ be the predictable σ-algebra on $[0,T] \times \Omega$.
For each $T > 0$ and $p > 1$, we use the following notation:

- $L^p(\mathcal{F}_T)$ is the set of random variables ξ which are $\mathcal{F}_T$-measurable and p-integrable.
- $I\!H^{p,T}$ is the set of real-valued predictable processes ϕ such that

$$\|\phi\|^p_{I\!H^{p,T}} := E\left[(\int_0^T \phi_t^2 dt)^{\frac{p}{2}}\right] < \infty.$$

  For $\beta > 0$ and $\phi \in I\!H^{2,T}$, we introduce the norm $\|\phi\|^2_{\beta,T} := E[\int_0^T e^{\beta s}\phi_s^2 ds]$.

- L^p_ν is the set of Borelian functions $\ell : \mathbf{R}^* \to \mathbf{R}$ such that $\int_{\mathbf{R}^*} |\ell(u)|^p \nu(du) < +\infty$.

  The set L^2_ν is a Hilbert space equipped with the scalar product

$$\langle \delta,\, \ell \rangle_\nu := \int_{\mathbf{R}^*} \delta(u)\ell(u)\nu(du) \quad \text{for all } \delta,\, \ell \in L^2_\nu \times L^2_\nu,$$

  and the norm $\|\ell\|^2_\nu := \int_{\mathbf{R}^*} |\ell(u)|^2 \nu(du) < +\infty$.

- $I\!H^{p,T}_\nu$ is the set of processes l which are *predictable*, that is, measurable

$$l : ([0,T] \times \Omega \times \mathbf{R}^*,\ \mathcal{P} \otimes \mathcal{B}(\mathbf{R}^*)) \to (\mathbf{R}\,, \mathcal{B}(\mathbf{R})); \quad (\omega, t, u) \mapsto l_t(\omega, u)$$

  such that

$$\|l\|^p_{I\!H^{p,T}_\nu} := E\left[(\int_0^T \|l_t\|^2_\nu\, dt)^{\frac{p}{2}}\right] < \infty.$$

  For $\beta > 0$ and $l \in I\!H^{2,T}_\nu$, we set $\|l\|^2_{\nu,\beta,T} := E[\int_0^T e^{\beta s} \|l_s\|^2_\nu\, ds]$.

- $S^{p,T}$ is the set of real-valued RCLL adapted processes ϕ such that

$$\|\phi\|^p_{S^p} := E(\sup_{0\le t\le T} |\phi_t|^p) < \infty.$$

When T is fixed and there is no ambiguity, we denote $I\!H^p$ instead of $I\!H^{p,T}$, $I\!H^p_\nu$ instead of $I\!H^{p,T}_\nu$, S^p instead of $S^{p,T}$.

- $\mathcal{T}_0$ denotes the set of stopping times τ such that $\tau \in [0,T]$ a.s.

Definition 2.1 (Driver, Lipschitz driver) *A function f is said to be a* driver *if*

- $f : [0,T] \times \Omega \times \mathbf{R}^2 \times L^2_\nu \to \mathbf{R}$
  $(\omega, t, x, \pi, \ell(\cdot)) \mapsto f(\omega, t, x, \pi, \ell(\cdot))$ *is* $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}(L^2_\nu)-$ *measurable,*
- $f(.,0,0,0) \in I\!H^2$.

A driver f is called a Lipschitz driver *if moreover there exists a constant $C \ge 0$ such that $dP \otimes dt$-a.s. , for each (x_1, π_1, ℓ_1), (x_2, π_2, ℓ_2),*

$$|f(\omega, t, x_1, \pi_1, \ell_1) - f(\omega, t, x_2, \pi_2, \ell_2)| \le C(|x_1 - x_2| + |\pi_1 - \pi_2| + \|\ell_1 - \ell_2\|_\nu).$$

Definition 2.2 (BSDE with jumps) *A solution of a BSDE with jumps with terminal time T, terminal condition ξ and driver f consists of a triple of processes (X, π, l) satisfying*

$$-dX_t = f(t, X_{t^-}, \pi_t, l_t(\cdot))dt - \pi_t dW_t - \int_{\mathbf{R}^*} l_t(u)\tilde{N}(dt, du); \qquad X_T = \xi. \quad (2.1)$$

where X is a RCLL optional process, and π (resp. l) is an $\mathbf{R}$-valued predictable process defined on $\Omega \times [0,T]$ (resp. $\Omega \times [0,T] \times \mathbf{R}^*$) such that the stochastic integral with respect to W (resp. $\tilde{N}$) is well defined.

Note that the process $f(t, X_{t^-}, \pi_t, l_t(\cdot))$ is predictable and satisfies $f(t, X_{t^-}, \pi_t, l_t(\cdot)) = f(t, X_t, \pi_t, l_t(\cdot))$ $dP \otimes dt$-a.s.

We recall the existence and uniqueness result for BSDEs with jumps established by Tang and Li (1994) in [26].

Theorem 2.3 (Existence and uniqueness) *Let $T > 0$. For each Lipschitz driver f, and each terminal condition $\xi \in L^2(\mathcal{F}_T)$, there exists a unique solution $(X, \pi, l) \in S^{2,T} \times I\!H^{2,T} \times I\!H^{2,T}_\nu$ of the BSDE with jumps (2.1).*

Remark 2.1 *We also provide an existence and uniqueness result in $S^{p,T} \times I\!H^{p,T} \times I\!H^{p,T}_\nu$ when f and ξ are p-integrable, with $p > 2$ (see Proposition A.5 in the Appendix). This result will be used in the study of dynamic risk measures in the case of ambiguity on the model (see the proof of Theorem 5.1).*

3 Linear BSDEs with jumps

We now focus on linear BSDEs with jumps which play a crucial role in the study of properties of general BSDEs.

We first provide some useful properties of exponential local martingales driven by a Brownian motion and a Poisson random measure.

3.1 Some properties of exponential local martingales

Let (β_t) be an $\mathbf{R}$-valued predictable process, a.s. integrable with respect to dW_t. Let $(\gamma_t(.))$ be an $\mathbf{R}$-valued *predictable* process defined on $[0,T]\times\Omega\times\mathbf{R}^*$, that is, $\mathcal{P}\otimes\mathcal{B}(\mathbf{R}^*)$-measurable, and a.s. integrable with respect to $\tilde{N}(ds,du)$.

Let $M=(M_t)_{0\leq t\leq T}$ be a local martingale given by

$$M_t := \int_0^t \beta_s dW_s + \int_0^t \int_{\mathbb{R}^*} \gamma_s(u)\tilde{N}(ds,du). \tag{3.2}$$

Let $Z=(Z_t)_{0\leq t\leq T}$ be the solution of

$$dZ_s = Z_{s^-}dM_s; \quad Z_0 = 1.$$

The process Z is the so-called exponential local martingale associated with the local martingale M, denoted by $\mathcal{E}(M)$. It is given by the Doléans-Dade formula (see (A.68) in the Appendix) :

$$\mathcal{E}(M)_s = \exp\{\int_0^s \beta_u dW_u - \frac{1}{2}\int_0^s \beta_u^2 du - \int_0^s \int_{\mathbb{R}^*} \gamma_r(u)\nu(du)dr\} \prod_{0<r\leq s}(1+\gamma_r(\Delta Y_r)) \tag{3.3}$$

where $Y_t := \int_{\mathbb{R}^*} uN([0,t],du)$. Classicaly, if $\gamma_t(\Delta Y_t)\geq -1$, $0\leq t\leq T$ a.s. then we have $\mathcal{E}(M)_t\geq 0$, $0\leq t\leq T$ a.s. Note that this property still holds for general exponential local martingales (see Appendix A). Since here M is driven by a Brownian motion and a Poisson random measure, we have more precisely the following property:

Proposition 3.1 *Let (β_t) and $(\gamma_t(.))$ be predictable $\mathbb{R}$-valued processes and let M be the local martingale defined by (3.2). The following assertions are equivalent:*

- *(i) For each $n\in I\!N$, $\gamma_{T_n}(\Delta Y_{T_n})\geq -1$ P-a.s., where $(T_n)_{n\in I\!N}$ is the increasing sequence of stopping times corresponding to the jumps times of Y.*
- *(ii) $\gamma_t(u)\geq -1$ $dP\otimes dt\otimes d\nu(u)$-a.s.*

Moreover, if one of this condition is satisfied, then we have $\mathcal{E}(M)_t \geq 0$, $0 \leq t \leq T$ *a.s.*

Similarly, if $\gamma_t(u) > -1$ $dP \otimes dt \otimes d\nu(u)$-a.s. , then, for each t, $\mathcal{E}(M)_t > 0$ a.s.
These precisions will be useful in the sequel, in particular to prove Theorem 5.1.

Proof. For each $s > 0$, we have $\prod_{0<r\leq s}(1+\gamma_r(\Delta Y_r)) = \prod_{n\in I\!N,\, 0<T_n\leq s}(1+\gamma_{T_n}(\Delta Y_{T_n}))$.
Hence, by formula (3.3), condition (i) implies that for each s, $\mathcal{E}(M)_s \geq 0$ a.s.
It remains to show that (i) is equivalent to (ii). Now, we have

$$E[\sum_{n\in I\!N} \mathbf{1}_{\{\gamma_{T_n}(\Delta Y_{T_n})<-1\}}] = E[\int_{\mathbf{R}^*\times\mathbf{R}_+} \mathbf{1}_{\{\gamma_r(u)<-1\}} N(du,dr)] = E[\int_{\mathbf{R}^*\times\mathbf{R}_+} \mathbf{1}_{\{\gamma_r(u)<-1\}} \nu(du)dr],$$

because $\nu(du)dt$ is the predictable compensator of the Poisson random measure $N(du, dt)$. The result follows. □

We now provide a sufficient condition for the square integrability property of $\mathcal{E}(M)$.

Proposition 3.2 *Let* (β_t) *and* $(\gamma_t(.))$ *be predictable* $\mathbb{R}$*-valued processes and let* M *be the local martingale defined by (3.2). Suppose that*

$$\int_0^T \beta_s^2 ds + \int_0^T \|\gamma_s\|_\nu^2 ds \tag{3.4}$$

is bounded. Then, we have $E[\mathcal{E}(M)_T^2] < +\infty$.

Note that in this case, by martingale inequalities, $(\mathcal{E}(M)_s)_{0\leq t\leq T} \in S^{2,T}$.

Proof. By the product formula (see (A.69)) (or by using the Doléans-Dade formula (3.3)), we get

$$\mathcal{E}(M)^2 = \mathcal{E}(2M + [M,M]),$$

where $[M,M]_t = \int_0^t \beta_s^2 ds + \sum_{s\leq t} \gamma_s^2(\Delta Y_s)$. Now,

$$\sum_{s\leq t} \gamma_s^2(\Delta Y_s) = \int_0^t \int_{\mathbb{R}^*} \gamma_s^2(u) N(ds,du) = \int_0^t \|\gamma_s\|_\nu^2\, ds + \int_0^t \int_{\mathbb{R}^*} \gamma_s^2(u) \tilde{N}(ds,du).$$

It follows that

$$\mathcal{E}(M)^2 = \mathcal{E}(N + \int_0^{\cdot} \beta_s^2 ds + \int_0^{\cdot} \|\gamma_s\|_\nu^2\, ds) = \mathcal{E}(N) \exp\{\int_0^{\cdot} \beta_s^2 ds + \int_0^{\cdot} \|\gamma_s\|_\nu^2\, ds\}, \tag{3.5}$$

where $N_t := 2M_t + \int_0^t \int_{\mathbb{R}^*} \gamma_s^2(u) \tilde{N}(ds, du)$. Note that N is a local martingale. Now, by assumption, there exists $K > 0$ such that $\exp\{ \int_0^T \beta_s^2 ds + \int_0^T \|\gamma_s\|_\nu^2\, ds\} \leq K$ a.s. Also, by (3.5), $\mathcal{E}(N)$ is nonnegative. Since it is also a local martingale, it follows that it is a supermartingale. Hence, we have $E[\mathcal{E}(M)_T^2] \leq E[\mathcal{E}(N)_T]\, K \leq K$, which ends the proof.
□

Remark 3.2 *For example, if β is bounded and if there exists $\psi \in L_\nu^2$ such that $dt \otimes dP \otimes d\nu(u)$-a.s. $|\gamma_t(u)| \leq \psi(u)$, then the process $\|\gamma_t\|_\nu^2$ is bounded. The random variable (3.4) is thus bounded, and hence, by the above proposition, $\mathcal{E}(M)_T \in L^2$. This property will be used in the study of linear BSDEs as well as in the comparison theorem (Theorem 4.2).*

More generally, we have the following property: if $\psi \in L_\nu^p$ for all $p \geq 2$, then $\mathcal{E}(M)_T$ is p-integrable for all $p \geq 2$. This property, as well as additional ones, are shown in the Appendix (see Proposition A.1). It will be used in Section 5.3, to solve a robust optimization problem, where some p-integrability conditions, with $p > 2$, are required.

3.2 Properties of linear BSDEs with jumps

We now show that the solution of a linear BSDE with jumps can be written as a conditional expectation via an exponential semimartingale.

Let (δ_t) and (β_t) be $\mathbf{R}$-valued predictable processes, supposed to be a.s. integrable with respect to dt and dW_t. Let $(\gamma_t(.))$ be a predictable $\mathbf{R}$-valued process defined on $[0,T] \times \Omega \times \mathbf{R}^*$, supposed to be a.s. integrable with respect to $\tilde{N}(ds, du)$.

For each $t \in [0,T]$, let $(\Gamma_{t,s})_{s \in [t,T]}$ be the unique solution of the following forward SDE

$$d\Gamma_{t,s} = \Gamma_{t,s^-} \left[\delta_s ds + \beta_s dW_s + \int_{\mathbb{R}^*} \gamma_s(u) \tilde{N}(ds, du) \right];\ \ \Gamma_{t,t} = 1. \tag{3.6}$$

The process $\Gamma_{t,.}$ can be written $\Gamma_{t,s} = e^{\int_t^s \delta_u du} Z_{t,s}$, where $(Z_{t,s})_{s \in [t,T]}$ is the solution of the following SDE

$$dZ_{t,s} = Z_{t,s^-} \left[\beta_s dW_s + \int_{\mathbb{R}^*} \gamma_s(u) \tilde{N}(ds, du) \right];\ \ Z_{t,t} = 1.$$

Theorem 3.3 *Let (δ, β, γ) be a bounded predictable process. Let Γ be the so-called* adjoint *process defined as the solution of SDE (3.6). Suppose that $\Gamma \in S^2$.*
Let (X_t, π_t, l_t) be the solution in $S^{2,T} \times I\!H^{2,T} \times I\!H_\nu^{2,T}$ of the linear BSDE

$$-dX_t = (\varphi_t + \delta_t X_t + \beta_t \pi_t + \langle \gamma_t, l_t \rangle_\nu) dt - \pi_t dW_t - \int_{\mathbb{R}^*} l_t(u) \tilde{N}(dt, du); \quad X_T = \xi. \tag{3.7}$$

The process (X_t) satisfies

$$X_t = E\left[\Gamma_{t,T}\ \xi + \int_t^T \Gamma_{t,s}\, \varphi(s) ds \mid \mathcal{F}_t\right], \quad 0 \le t \le T, \quad \text{a.s.}, \tag{3.8}$$

Proof. Fix $t \in [0, T]$. To simplify notation, let us denote $\Gamma_{t,s}$ by Γ_s for $s \in [t, T]$. By the Itô product formula, and denoting $\Gamma_{t,s}$ by Γ_s for $s \in [t, T]$, we have:

$$\begin{aligned}
-d(X_s \Gamma_s) &= -X_{s^-} d\Gamma_s - \Gamma_{s^-} dX_s - d[X, \Gamma]_s \\
&= -X_s \Gamma_s \delta_s ds + \Gamma_s \left[\varphi_s + \delta_s X_s + \beta_s \pi_s + \langle \gamma_s, l_s \rangle_\nu\right] ds \\
&\quad - \beta_s \pi_s \Gamma_s ds - \Gamma_s \langle \gamma_s, l_s \rangle_\nu ds - \Gamma_s (X_s \beta_s + \pi_s) dW_s - \Gamma_{s^-} \int_{\mathbb{R}} l_s(u)(1 + \gamma_s(u)) \tilde{N}(du, ds) \\
&= \Gamma_s \varphi_s ds - dM_s,
\end{aligned}$$

with $dM_s = -\Gamma_s (X_s \beta_s + \pi_s) dW_s - \Gamma_{s^-} \int_{\mathbb{R}} l_s(u)(1 + \gamma_s(u)) \tilde{N}(du, ds)$. By integrating between t and T, we get

$$X_t - \xi \Gamma_{t,T} = \int_t^T \Gamma_{t,s} \varphi_s ds - M_T + M_t \quad \text{a.s.} \tag{3.9}$$

Recall that $\Gamma_{t,.} \in S^2$ and that $X \in S^2$, $\pi \in I\!H^2$ and $l \in I\!H^2_\nu$. Moreover, the processes δ, β and γ are bounded. It follows that the local martingale M is an uniformly martingale. Hence, by taking the conditional expectation in equality (3.9), we get equality (3.8). □

This property together Proposition 3.1 yields the following corollary, which will be used to prove the comparison theorems.

Corollary 3.4 *Suppose that the assumptions of Theorem 3.3 are satisfied.*

- *Suppose that the inequality $\gamma_t(u) \ge -1$ holds $dP \otimes dt \otimes d\nu(u)$-a.s.*
  If $\varphi_t \ge 0$, $t \in [0, T]$, $dP \otimes dt$ a.s. and $\xi \ge 0$ a.s., then $X_t \ge 0$ a.s. for all $t \in [0, T]$.

- *Suppose that the inequality $\gamma_t(u) > -1$ holds $dP \otimes dt \otimes d\nu(u)$-a.s. If $\varphi_t \geq 0$, $t \in [0,T]$, $dP \otimes dt$ a.s. and $\xi \geq 0$ a.s., and if $X_{t_0} = 0$ a.s. for some $t_0 \in [0,T]$, then $\varphi_t = 0$ $dP \otimes dt$-a.s. on $[t_0,T]$, and $\xi = 0$ a.s on A.*

Proof. Let us prove the first assertion. Since $\gamma_t(u) \geq -1$ $dP \otimes dt \otimes d\nu(u)$-a.s., by Proposition 3.1, we get $\Gamma_{t,T} \geq 0$ a.s. and the result follows from the representation formula for linear BSDEs (3.8).
The second assertion follows from similar arguments and the fact that if $\gamma_t(u) > -1$ $dP \otimes dt \otimes d\nu(u)$-a.s., then $\Gamma_{t,T} > 0$ a.s. □

Note that when $\xi \geq 0$ and $\varphi \geq 0$, if the process γ can take values < -1 with positive measure, then the solution X of the linear BSDE may take strictly negative values.
moreover, suppose that $\xi \geq 0$ and $\varphi \geq 0$ a.s. and that the process $\gamma \geq -1$ a.s. but can take the value -1 with positive measure. Then, in general, the equality $X_0 = 0$ does not imply that $\xi = 0$ and $\varphi = 0$ a.s.

Example 3.1 *Consider the simple but instructive case, where γ is a real constant and $\delta = \beta = 0$, $\varphi = 0$. Suppose also that $\nu(du) := \delta_1(du)$, where δ_1 denotes Dirac measure at 1. Set $N_t := N([0,t] \times \{1\})$. The process N_t is then a Poisson process with parameter 1 and we have $\tilde{N}_t := \tilde{N}([0,t] \times \{1\}) = N_t - t$. Let f be the linear driver given by*

$$f(\ell) = \langle \gamma, \ell \rangle_\nu = \gamma \ell(1). \tag{3.10}$$

In this case, the adjoint process $\Gamma_{0,s}$, denoted for simplicity by Γ_s, satisfies

$$\Gamma_T = ((1+\gamma)e^{-\gamma})^{N_T} e^{\gamma \tilde{N}_T} = (1+\gamma)^{N_T} e^{-\gamma T}. \tag{3.11}$$

Let X be the solution of the BSDE associated with terminal condition $\xi := N_T$. By the representation property of linear BSDEs with jumps (see equality (3.8)), we have $X_0 = E[\Gamma_T \xi] = E[\Gamma_T N_T]$. Now, for each s, $E[N_T s^{N_T}] = sTe^{T(s-1)}$. Taking $s = 1+\gamma$, we thus obtain

$$X_0 = E[\Gamma_T N_T] = e^{-\gamma T} E[(1+\gamma)^{N_T} N_T] = (1+\gamma)T. \tag{3.12}$$

Similarly, for each $t \in [0,T]$, we get

$$X_t = E[\Gamma_{t,T} N_T \,|\, \mathcal{F}_t] = E[\Gamma_{t,T}(N_T - N_t) \,|\, \mathcal{F}_t] + E[\Gamma_{t,T} N_t \,|\, \mathcal{F}_t] = (1+\gamma)(T-t) + N_t.$$

Suppose now that

$$\gamma < -1 \quad \text{(still with} \quad \xi = N_T). \tag{3.13}$$

By equality (3.12), we have $X_0 = (1+\gamma)T < 0$ *although* $\xi = N_T \geq 0$ *a.s. We also have for each time* t, $P(X_t < 0) > 0$.

Let us now consider the following example, with the same linear driver (3.10), but with

$$\gamma = -1 \quad \text{and} \quad \xi = \mathbf{1}_{\{T_1 \leq T\}}. \tag{3.14}$$

By equality (3.11), it follows that $\Gamma_T \geq 0$ *a.s. and also that* $\Gamma_T = 0$ *a.s. on* $\{N_T \geq 1\} = \{T_1 \leq T\}$. *Hence,* $X_0 = E[\Gamma_T \xi] = E[\Gamma_T \mathbf{1}_{\{T_1 \leq T\}}] = 0$ *although* $\xi \geq 0$ *a.s. and* $P(\xi > 0) = P(T_1 \leq T) > 0$.

We now provide a second corollary of Theorem 3.3 which will be used to prove the second optimization principle for BSDEs with jumps (Theorem 4.6).

Corollary 3.5 *Suppose that assumptions of Theorem 3.3 are satisfied. If* $\xi \geq 0$ *a.s. and* $\varphi_t \geq -\varepsilon \, dP \otimes dt$ *a.s. for some* $\varepsilon \geq 0$, *then* $X_t \geq -\varepsilon \, T e^{CT}$ *a.s. for all* $t \in [0,T]$, *where* C *is a bound of* (δ_t).

Proof. By Theorem 3.3, we have

$$X_t \geq E\left[\int_t^T \Gamma_{t,s}\varphi_s ds \mid \mathcal{F}_t\right] \geq -\varepsilon E\left[\int_t^T e^{\int_t^s \delta_u du} Z_{t,s} ds \mid \mathcal{F}_t\right], \quad 0 \leq t \leq T, \quad \text{a.s.}$$

Since $Z_{t,.}$ is a martingale (defined by (3.3) and since $|\delta_t| \leq C$, we get $X_t \geq -\varepsilon T e^{CT}$ a.s. □

4 Comparison theorems and optimization principles for BSDEs with jumps

4.1 Comparison theorems

The comparison theorems are key properties of BSDEs and play a crucial role in the study of optimization problems expressed in terms of BSDEs. In [25], Royer established a comparison theorem and a strict comparison theorem for BSDEs with jumps. Here, we prove these theorems under less restrictive hypotheses and provide some optimization principles for BSDEs with jumps.

We begin by a preliminary result which will be used to prove the comparison theorems.

Lemma 4.1 (Comparison result with respect to a linear BSDE) *Let (δ, β, γ) be a bounded predictable process and for each t, let $\Gamma_{t,\cdot}$ be the exponential semimartingale solution of SDE (3.6). Suppose that*

$$\Gamma_{t,\cdot} \in S^2 \quad \forall t \qquad \text{and} \qquad \gamma_t(u) \geq -1 \quad dP \otimes dt \otimes \nu(du) - \text{a.s.}$$

Let $\xi \in L^2(\mathcal{F}_T)$ and h be a driver (non necessarily Lipschitz). Let (X_t, π_t, l_t) be a solution in $S^{2,T} \times I\!H^{2,T} \times I\!H_\nu^{2,T}$ of the BSDE

$$-dX_t = h(t, X_t, \pi_t, l_t(\cdot))dt - \pi_t dW_t - \int_{\mathbb{R}^*} l_t(u)\tilde{N}(dt, du)\ ;\ X_T = \xi. \quad (4.15)$$

Let $\varphi \in I\!H^{2,T}$. Suppose that

$$h(t, X_t, \pi_t, l_t) \geq \varphi_t + \delta_t X_t + \beta_t \pi_t + \langle \gamma_t\,,\, l_t \rangle_\nu, \quad 0 \leq t \leq T, \quad dP \otimes dt - \text{a.s.} \quad (4.16)$$

Then, X is a.s. greater or equal to the solution given by (3.8) of the linear BSDE (3.7). In other terms,

$$X_t \geq E\left[\Gamma_{t,T}\, \xi + \int_t^T \Gamma_{t,s}\, \varphi(s) ds \mid \mathcal{F}_t\right], \quad 0 \leq t \leq T, \quad \text{a.s.} \quad (4.17)$$

Proof. Fix $t \in [0, T]$. Since $\gamma_t(u) \geq -1$ $dP \otimes dt \otimes \nu(du)$-a.s. , it follows that $\Gamma_{t,\cdot} \geq 0$ a.s. To simplify notation, let us denote $\Gamma_{t,s}$ by Γ_s for $s \in [t, T]$. By the Itô product formula, and denoting $\Gamma_{t,s}$ by Γ_s for $s \in [t, T]$, we have:

$$\begin{aligned} -d(X_s \Gamma_s) = {} & -X_{s^-}\Gamma_{s^-}[\delta_s ds + \beta_s dW_s + \int_{\mathbb{R}} \gamma_s(u)\tilde{N}(du, ds)] + \Gamma_{s^-} h(s, X_s, \pi_s, l_s) ds \\ & - \Gamma_{s^-}[\pi_s dW_s + \int_{\mathbb{R}} l_s(u)\tilde{N}(du, ds)] - \pi_s \Gamma_s \beta_s ds - \Gamma_{s^-} \int_{\mathbb{R}} \gamma_s(u) l_s(u) N(du, ds). \end{aligned}$$

Using inequality (4.16) together with the non negativity of Γ, and doing the same computations as in the proof of Theorem 3.3, we derive that

$$-d(X_s \Gamma_s) \geq \Gamma_s \varphi_s ds - dM_s,$$

where M is a martingale (since $\Gamma_{t,\cdot} \in S^2$ and since (δ_t) and (β_t) are bounded). By integrating between t and T and by taking the conditional expectation, we derive inequality (4.17). Now, by Theorem 3.3, the second member of this inequality corresponds to the solution of the linear BSDE (3.7). The proof is thus complete. □

The comments made in the linear case (see in particular example (3.13)) show the relevance of the assumption $\gamma_t(u) \geq -1$ in the above lemma.

Note also that if δ, β, γ are bounded and if $|\gamma_t| \leq \psi$, where $\psi \in L^2_\nu$, Proposition 3.2 yields that, for each t, $\Gamma_{t,\cdot} \in S^2$. Using this remark together with the above lemma, we now show the general comparison theorems for BSDEs with jumps.

Theorem 4.2 (Comparison theorem for BSDEs with jumps) *Let ξ_1 and $\xi_2 \in L^2(\mathcal{F}_T)$. Let f_1 be a Lipschitz driver. Let f_2 be a driver. For $i = 1, 2$, let (X^i_t, π^i_t, l^i_t) be a solution in $S^{2,T} \times I\!H^{2,T} \times I\!H^{2,T}_\nu$ of the BSDE*

$$-dX^i_t = f_i(t, X^i_t, \pi^i_t, l^i_t)dt - \pi^i_t dB_t - \int_{\mathbb{R}^*} l^i_t(u)\tilde{N}(dt, du); \quad X^i_T = \xi_i. \tag{4.18}$$

Assume that there exists a bounded predictable process (γ_t) such that $dt \otimes dP \otimes \nu(du)$-a.s.,

$$\gamma_t(u) \geq -1 \quad and \quad |\gamma_t(u)| \leq \psi(u), \tag{4.19}$$

where $\psi \in L^2_\nu$, and such that

$$f_1(t, X^2_t, \pi^2_t, l^1_t) - f_1(t, X^2_t, \pi^2_t, l^2_t) \geq \langle \gamma_t\, , \, l^1_t - l^2_t \rangle_\nu, \;\; t \in [0, T], \;\; dt \otimes dP \;\; a.s.. \tag{4.20}$$

Assume that

$$\xi_1 \geq \xi_2 \;\; a.s. \tag{4.21}$$

and

$$f_1(t, X^2_t, \pi^2_t, l^2_t) \geq f_2(t, X^2_t, \pi^2_t, l^2_t), \;\; t \in [0, T], \;\; dt \otimes dP \;\; a.s.. \tag{4.22}$$

Then, we have

$$X^1_t \geq X^2_t \;\; a.s. \; for \; all \; t \in [0, T]. \tag{4.23}$$

Moreover, if inequality (4.22) is satisfied for (X^1_t, π^1_t, l^1_t) instead of (X^2_t, π^2_t, l^2_t) and if f_2 (instead of f_1) is Lipschitz and satisfies (4.20), *then inequality (4.23) still holds.*

Proof. Put $\bar{X}_t = X^1_t - X^2_t$; $\bar{\pi}_t = \pi^1_t - \pi^2_t$; $\bar{l}_t(u) = l^1_t(u) - l^2_t(u)$. Then

$$-d\bar{X}_t = h_t dt - \bar{\pi}_t dW_t - \int_{\mathbb{R}^*} \bar{l}_t(u)\tilde{N}(dt, du); \quad \bar{X}_T = \xi_1 - \xi_2,$$

where $h_t := f_1(t, X^1_t, \pi^1_t, l^1_t) - f_2(t, X^2_t, \pi^2_t, l^2_t)$. The proof now consists to show that there exists δ and β such that h_t satisfies inequality (4.16) and then to apply the comparison result with respect to a linear BSDE (Lemma 4.1).

We have

$$\begin{aligned} h_t = {} & f_1(t, X^1_t, \pi^1_t, l^1_t) - f_1(t, X^2_t, \pi^1_t, l^1_t) + f_1(t, X^2_t, \pi^1_t, l^1_t) - f_1(t, X^2_t, \pi^2_t, l^1_t)) \\ & + f_1(t, X^2_t, \pi^2_t, l^1_t) - f_1(t, X^2_t, \pi^2_t, l^2_t) + f_1(t, X^2_t, \pi^2_t, l^2_t) - f_2(t, X^2_t, \pi^2_t, l^2_t). \end{aligned} \tag{4.24}$$

Let $\varphi_t := f_1(t, X^2_{t^-}, \pi^2_t, l^2_t) - f_2(t, X^2_{t^-}, \pi^2_t, l^2_t)$ and

$$\delta_t := \frac{f_1(t, X^1_{t^-}, \pi^1_t, l^1_t) - f_1(t, X^2_{t^-}, \pi^1_t, l^1_t)}{\bar{X}_t} \mathbf{1}_{\{\bar{X}_t \neq 0\}}$$
$$\beta_t := \frac{f_1(t, X^2_{t^-}, \pi^1_t, l^1_t) - f_1(t, X^2_{t^-}, \pi^2_t, l^1_t)}{\bar{\pi}_t} \mathbf{1}_{\{\bar{\pi}_t \neq 0\}}.$$

By the assumption (4.20) on f_1, we get

$$h_t \geq \varphi_t + \delta_t \bar{X}_t + \beta_t \bar{\pi}_t + \langle \gamma_t \,,\, \bar{l}_t \rangle_\nu \quad dt \otimes dP - \text{a.s.}$$

Since f_1 is Lipschitz, the predictable processes (δ_t) and (β_t) are bounded. By Assumption (4.19), it follows from Proposition 3.2 that for each t, $\Gamma_{t,.} \in S^2$, where the process $\Gamma_{t,.}$ is defined by (3.6). The result then follows from the comparison theorem with respect to a linear BSDE (see Lemma 4.1).

Since $\gamma_t(u) \geq -1$, it follows that $\Gamma_{t,.} \geq 0$ a.s. By the comparison result with respect to a linear BSDE (see Lemma 4.1), we thus derive that

$$\bar{X}_t \geq E\left[\Gamma_{t,T}\ (\xi_1 - \xi_2) + \int_t^T \Gamma_{t,s}\, \varphi(s) ds \mid \mathcal{F}_t\right], \quad 0 \leq t \leq T, \quad \text{a.s.} \tag{4.25}$$

Now, by assumption, $\varphi_t \geq 0$ $dP \otimes dt$-a.s. and $\xi_1 - \xi_2 \geq 0$ a.s. It follows that for all $t \in [0, T]$, $\bar{X}_t \geq 0$ a.s. , which gives $X^1_t \geq X^2_t$ a.s. The proof of the first assertion is thus complete.

The second assertion follows from the same arguments but linearizing f^2 instead of f^1. □

Remark 4.3 *Note that the presence of jumps as well as inequality (4.20), which is a relatively weak assumption, do not really allow us to proceed with a linearization method as in the Brownian case (see [10]). More precisely, in equality (4.24), we cannot deal with the increement of f^1 with respect to l as with those with respect to x or π. Indeed, there is somehow an asymmetry between the role of negative jumps and that of positive ones of $\bar{X}$. The above lemma thus appears as a preliminary step before proving the general comparison theorem in the case of jumps.*

Note also that the second condition in (4.19) on γ_t could be replaced by the weaker, but less tractable, assumption $\Gamma \in S^2$, where Γ is defined in (3.6).

Corollary 4.3 (Non negativity sufficient condition) *Let $\xi \geq 0 \in L^2(\mathcal{F}_T)$ and f be a Lipschitz driver such that $f(t, 0, 0, 0) \geq 0$ $dt \otimes dP$ a.s.*
Let (X, π, l) be the solution of the BSDE associated with driver f and terminal condition ξ. Suppose that there exits (γ_t) satisfying condition (4.19) *as well as the inequality*

$$f(t, 0, 0, l_t) - f(t, 0, 0, 0) \geq \langle \gamma_t \,,\, l_t \rangle_\nu, \quad t \in [0, T], \quad dt \otimes dP \ \ a.s. \tag{4.26}$$

Then, $X_t \geq 0$ a.s. $t \in [0, T]$.

Proof. Apply Theorem 4.2 with $f^1 = f$, $f^2 = 0$, $\xi_1 = \xi$, and $\xi_2 = 0$. $\square$

We now provide a strict comparison theorem, which holds under an additional assumption. This property is an important tool for the study of optimization problems expressed in terms of BSDEs since it allows us to obtain a necessary condition of optimality (see Proposition 4.1 assertion 2.).

Theorem 4.4 (Strict comparison theorem) *Suppose that the assumptions of Theorem 4.2 hold and that the inequality $\gamma_t(u) > -1$ holds $dt \otimes dP \otimes d\nu(u)$-a.s.*
If $X^1_{t_0} = X^2_{t_0}$ a.s. on A for some $t_0 \in [0, T]$ and $A \in \mathcal{F}_{t_0}$, then $X^1_\cdot = X^2_\cdot$ a.s. on $[t_0, T] \times A$, $\xi_1 = \xi_2$ a.s. on A and (4.22) holds as an equality in $[t_0, T] \times A$.

Proof. The result follows from inequality (4.25) and the second assertion of Corollary 3.4. $\square$

The example (3.14) given in section 3 shows the relevance of the assumption $\gamma_t(u) > -1$ in the strict comparison theorem.

Note also that the conditions under which we have proved the comparison theorems are weaker that the one stated in the literature, in particular in [25] (see some more details in Section 4.3).

4.2 Optimization principles

From the comparison theorem, we derive optimization principles for *minima* of BSDEs which generalize those of El Karoui et al. (1997) to the case of jumps.

Theorem 4.5 (Optimization principle I) *Let ξ in $L^2(\mathcal{F}_T)$ and let $(f, f^\alpha; \alpha \in \mathcal{A})$ be a family of Lipschitz drivers. Let (X, π, l) (resp. $(X^\alpha, \pi^\alpha, l^\alpha)$) be the solution of the BSDE associated to terminal condition ξ and driver f (resp. f^α).*

Suppose that there exists a parameter $\bar{\alpha} \in \mathcal{A}$ such that

$$f(t, X_t, \pi_t, l_t) = \operatorname*{ess\,inf}_\alpha f^\alpha(t, X_t, \pi_t, l_t) = f^{\bar{\alpha}}(t, X_t, \pi_t, l_t),\ 0 \leq t \leq T,\ \ dP \otimes dt-\text{a.s.} \tag{4.27}$$

and that for each $\alpha \in \mathcal{A}$, there exists a predictable process γ^α satisfying (4.19) and

$$f^\alpha(t, X_t, \pi_t, l^\alpha_t) - f^\alpha(t, X_t, \pi_t, l_t) \geq \langle \gamma^\alpha_t, l^\alpha_t - l_t \rangle_\nu,\ \ t \in [0, T],\ \ dt \otimes dP \ \ a.s.. \tag{4.28}$$

Then,

$$X_t = \operatorname*{ess\,inf}_{\alpha} X_t^{\alpha} = X_t^{\bar{\alpha}}, \;\; 0 \le t \le T, \text{ a.s.} \tag{4.29}$$

Proof. For each α, since $f(t, X_t, \pi_t, l_t) \le f^{\alpha}(t, X_t, \pi_t, l_t)\ dP \otimes dt$-a.s. , the comparison Theorem 4.2 gives that $X_t \le X_t^{\alpha}$, $0 \le t \le T$, P-a.s. It follows that

$$X_t \le \operatorname*{ess\,inf}_{\alpha} X_t^{\alpha} \qquad 0 \le t \le T,\; P - a.s. \tag{4.30}$$

By assumption, X_t is the solution of the BSDE associated with $f^{\bar{\alpha}}$. By uniqueness of the solution of the Lipschitz BSDE associated to $f^{\bar{\alpha}}$, we derive that $X_t = X_t^{\bar{\alpha}}$, $0 \le t \le T$, a.s. which implies that inequality in (4.30) is an equality. □

Remark 4.4 *If in* (4.27)*, the* $essinf$ *is replaced by esssup and if the inverse inequality holds in* (4.28)*, then* (4.29) *holds with an esssup instead of an* $essinf$*.*

Theorem 4.6 (Optimization principle II) . *Let ξ in $L^2(\mathcal{F}_T)$ and let $(f, f^{\alpha}; \alpha \in \mathcal{A})$ be a family of Lipschitz drivers. Suppose that the drivers f^{α}, $\alpha \in \mathcal{A}$ are equi-Lipschitz with common Lipschitz constant C. Let (X, π, l) be a solution of the BSDE associated to terminal condition ξ and driver f and $(X^{\alpha}, \pi^{\alpha}, l^{\alpha})$ be the solution of the BSDE associated with terminal condition ξ and driver f^{α}. Suppose that for each $\alpha \in \mathcal{A}$,*

$$f(t, X_t, \pi_t, l_t) \le f^{\alpha}(t, X_t, \pi_t, l_t), \;\; 0 \le t \le T, \;\; dP \otimes dt - \text{a.s.} \tag{4.31}$$

and that there exists γ^{α} and δ^{α} satisfying (4.19) *and*

$$\langle \gamma_t^{\alpha}, l_t^{\alpha} - l_t \rangle_{\nu} \le f^{\alpha}(t, X_t, \pi_t, l_t^{\alpha}) - f^{\alpha}(t, X_t, \pi_t, l_t) \le \langle \delta_t^{\alpha}, l^{\alpha} - l_t \rangle_{\nu}, \;\; t \in [0, T], \;\; dt \otimes dP \;\; a.s.. \tag{4.32}$$

Suppose also that for each $\varepsilon > 0$, there exists $\alpha^{\varepsilon} \in \mathcal{A}$ such that

$$f^{\alpha^{\varepsilon}}(t, X_t, \pi_t, l_t) - \epsilon \le f(t, X_t, \pi_t, l_t), \;\; 0 \le t \le T, \;\; dP \otimes dt - \text{a.s.} \tag{4.33}$$

Then,

$$X_t = \operatorname*{ess\,inf}_{\alpha} X_t^{\alpha}, \;\; 0 \le t \le T, \text{ a.s.}$$

Lemma 4.7 *Suppose that the assumptions of Theorem 4.2 are satisfied but with condition (4.22) replaced by*

$$f_1(t, X_t^2, \pi_t^2, l_t^2) \ge f_2(t, X_t^2, \pi_t^2, l_t^2) - \varepsilon, \;\; t \in [0, T], \;\; dt \otimes dP \;\; a.s. \tag{4.34}$$

We then have

$$X_t^1 \geq X_t^2 - \varepsilon\, T e^{CT} \ \textit{a.s. for all } t \in [0,T], \tag{4.35}$$

where C is the Lipschitz constant of f_1.

Moreover, if f^2 is Lipschitz (instead of f^1) with constant C and if Assumption (4.20) *is satisfied along (X_t^1, π_t^1, l_t^1) instead of (X_t^2, π_t^2, l_t^2), then inequality (4.35) still holds.*

Proof. The proof follows from Corollary 3.5 and inequality (4.25). □

Proof of Theorem 4.6 By the comparison theorem, $X_t \leq X_t^\alpha$, $0 \leq t \leq T$, a.s. Hence, $X_t \leq \operatorname{ess\,inf}_\alpha X_t^\alpha$ a.s. for each $t \in [0,T]$.
Let us now show the inverse inequality. By the second assertion of Lemma 4.7, we have

$$X_t \geq X_t^{\alpha^\varepsilon} - \varepsilon\, T e^{CT} \geq \operatorname*{ess\,inf}_\alpha X_t^\alpha - \varepsilon\, T e^{CT}, \quad 0 \leq t \leq T, \text{ a.s.}$$

Since this inequality holds for each $\varepsilon > 0$, we obtain

$$X_t \geq \operatorname*{ess\,inf}_\alpha X_t^\alpha, \quad 0 \leq t \leq T, \quad \text{a.s.}$$

Hence, this inequality is an equality. □

The optimization principle I together with the strict comparison theorem, yield the following sufficient (resp. necessary) condition of optimality.

Proposition 4.1 *Suppose that Assumptions of Theorem 4.5 (resp. 4.6) are satisfied. Let $\hat\alpha \in \mathcal{A}$ and let S be a stopping time smaller or equal to T.*

1. *Suppose that*

$$f(t, X_t, \pi_t, l_t) \;=\; f^{\hat\alpha}(t, X_t, \pi_t, l_t), \qquad S \leq t \leq T, \quad dP \otimes dt - \text{a.s.} \tag{4.36}$$

Then, $\hat\alpha$ is S-optimal (i.e. $\operatorname{ess\,inf}_\alpha X_S^\alpha = X_S^{\hat\alpha}$).

2. *(Necessary condition of optimality) Suppose $\hat\alpha$ is S-optimal and that the associated process $\gamma_t^{\hat\alpha}$ satisfies the strict inequality*

$$\gamma_t^{\hat\alpha} > -1 \quad dP \otimes dt \otimes d\nu - \text{a.s.}$$

Then, equality (4.36) holds.

4.3 Remarks on the assumptions of the comparison theorem

Let us introduce the following condition. Let $T > 0$.

Assumption 4.1 *A driver f is said to satisfy Assumption 4.1 if the following holds:*
$dP \otimes dt$-a.s for each $(x, \pi, l_1, l_2) \in [0, T] \times \Omega \times \mathbb{R}^2 \times (L^2_\nu)^2$,

$$f(t, x, \pi, l_1) - f(t, x, \pi, l_2) \geq \langle \theta_t^{x,\pi,l_1,l_2} \, , \, l_1 - l_2 \rangle_\nu,$$

with

$$\theta : [0, T] \times \Omega \times \mathbb{R}^2 \times (L^2_\nu)^2 \mapsto L^2_\nu \, ; \, (\omega, t, x, \pi, l_1, l_2) \mapsto \theta_t^{x,\pi,l_1,l_2}(\omega, .)$$

$\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}((L^2_\nu)^2)$-measurable, bounded, and satisfying $dP \otimes dt \otimes d\nu(u)$-a.s. , for each $(x, \pi, l_1, l_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$,

$$\theta_t^{x,\pi,l_1,l_2}(u) \geq -1 \quad \textit{and} \quad |\theta_t^{x,\pi,l_1,l_2}(u)| \leq \psi(u), \tag{4.37}$$

where $\psi \in L^2_\nu$.

This assumption is stronger than the one made in the comparison theorem (Theorem 4.2). Indeed, if the driver f_1 satisfies Assumption 4.1, then it satisfies condition (4.20) assumed in the comparison theorem, with $\gamma_t = \theta_t^{X^2_{t-},\pi^2_t,l^1_t,l^2_t}$. Note also that condition (4.20) is only required along (X^2_t, π^2_t, l^2_t) (the solution of the second BSDE) as well as l^1_t (the third coordinate of the solution of the first BSDE) but not necessarily for all (x, π, l).

Also, if f^α satisfies Assumption 4.1, then it satisfies the weaker condition (4.32) assumed in the optimization principle II.

The above assumption thus appears as much stronger than the one made in the comparison theorems. However, it ensures a monotony property with respect to terminal condition, in the following sense: for all ξ^1, $\xi^2 \in L^2(\mathcal{F}_T)$ with $\xi^1 \geq \xi^2$ a.s. , we have $X(\xi^1) \geq X(\xi^2)$ a.s. , where $X(\xi^1)$ (resp. $X(\xi^2)$) denotes the solution of the BSDE associated with f and ξ^1 (resp. ξ^2). This clearly follows from the comparison theorem applied to $f^1 = f^2 = f$. As we will see in the next section, this assumption will be appropriate to ensure the *monotonicity* property of a dynamic risk measure induced by a BSDE.

Remark 4.5 *Assumption 4.1 implies that for each (x, π, l_1, l_2),*

$$f(t, x, \pi, l_1) - f(t, x, \pi, l_2) \leq \langle \gamma_t^{x,\pi,l_1,l_2} \, , \, l_1 - l_2 \rangle_\nu \tag{4.38}$$

where $\gamma_t^{x,\pi,l_1,l_2}(u) = \theta_t^{x,\pi,l_2,l_1}(u)$*. Actually, Assumption 4.1 is equivalent to Assumption (4.38), where* γ *satisfies conditions (4.37).*

Note that Assumption 4.1, even if it is stronger than the one made in the comparison theorem (Theorem 4.2), is still weaker than the assumption made in Royer [25], where, in particular, it is moreover required that $\gamma_t^{x,\pi,l_1,l_2} \geq C_1$ *with* $C_1 > -1$.

5 Dynamic risk measures induced by BSDEs with jumps, robust optimization problems

5.1 Definitions and first properties

Let $T' > 0$ be a time horizon. Let f be a Lipschitz driver such that $f(\cdot, 0, 0, 0) \in I\!H^{2,T'}$.
We define the following functional: for each $T \in [0, T']$ and $\xi \in L^2(\mathcal{F}_T)$, set

$$\rho_t^f(\xi, T) = \rho_t(\xi, T) := -X_t(\xi, T), \quad 0 \leq t \leq T, \tag{5.39}$$

where $X_t(\xi, T)$ denotes the solution of the BSDE (2.1) with terminal condition ξ and terminal time T. If T represents a given maturity and ξ a financial position at time T, then $\rho_t(\xi, T)$ will be interpreted as the risk measure of ξ at time t. The functional $\rho : (\xi, T) \mapsto \rho_\cdot(\xi, T)$ defines then a dynamic risk measure induced by the BSDE with driver f.

We now provide properties of such a dynamic risk measure. We point out that, contrary to the Brownian case, the monotonicity property of ρ_t, which is naturally required for risk measures, is not automatically satisfied and needs the additional assumption Assumption 4.1, introduced in Section 4.3.

• **Consistency.** By the flow property (see (A.79) in the Appendix), ρ is *consistent* : more precisely, let $T \in [0, T']$ and let $S \in \mathcal{T}_{0,T}$ be a stopping time, then for each time t smaller than S, the risk-measure associated with position ξ and maturity T coincides with the risk-measure associated with maturity S and position $-\rho_S(\xi, T) = X_S(\xi, T)$, that is,
(CS) $\forall t \leq S$, $\rho_t(\xi, T) = \rho_t(-\rho_S(\xi, T), S)$ a.s.
Indeed, $\forall t \leq S$, $X_t(\xi, T) = X_t(X_S(\xi, T), S)$ a.s..

• **Continuity.** Let $T \in [0, T']$. Let $\{\theta^\alpha, \alpha \in \mathbf{R}\}$ be a family of stopping times in $\mathcal{T}_{0,T}$, converging a.s. to a stopping time $\theta \in \mathcal{T}_{0,T}$ as α tends to α_0. Let $(\xi^\alpha, \alpha \in \mathbf{R})$ be a family of random variables such that $\mathbb{E}[\text{ess sup}_\alpha (\xi^\alpha)^2] < +\infty$, and for each α, ξ^α is $\mathcal{F}_{\theta^\alpha}$-measurable. Suppose also that ξ^α converges a.s. to an $\mathcal{F}_\theta$-measurable random variable ξ as α tends to α_0. Then, for each $S \in \mathcal{T}_{0,T}$, the random variable $\rho_S(\xi^\alpha, \theta^\alpha) \to \rho_S(\xi, \theta)$ a.s. and the process $\rho(\xi^\alpha, \theta^\alpha) \to \rho(\xi, \theta)$ in $S^{2,T}$ when $\alpha \to \alpha_0$ (see Proposition A.7 in the Appendix).

• **Zero-one law.** If $f(t, 0, 0) = 0$, then the risk-measure associated to the null position is equal to 0. More precisely, the risk-measure satisfies the *Zero-one law* property:

(ZO) $\rho_t(\mathbf{1}_A\xi, T) = \mathbf{1}_A\rho_t(\xi, T)$ *a.s* for $t \leq T$, $A \in \mathcal{F}_t$, and $\xi \in L^2(\mathcal{F}_T)$.

• **Translation invariance.** If f does not depend on x, then the associated risk-measure satisfies the *translation invariance property*:

(TI) $\rho_t(\xi + \xi', T) = \rho_t(\xi, T) - \xi'$, for any $\xi \in L^2(\mathcal{F}_T)$ and $\xi' \in L^2(\mathcal{F}_t)$.

This situation can be interpreted as a market with an interest rate r_t equal to zero. The case $r_t \neq 0$ corresponds to a BSDE with a driver of the form $-r_t x + g(t, \pi, l)$ and can be reformulated as a problem with a driver independant of x by discounting the positions ξ (see Section 5.4.1). The general case when f depends on x in a nonlinear way may be interpreted in terms of ambiguity on the interest rate (see Section 5.4.2 and Remark **??**).

• **Homogeneous property.** If f is positively homogenous with respect to (x, π, l), then the risk-measure ρ is *positively homogenous* with respect to ξ, that is,

$$\rho_.(\lambda\xi, T) = \lambda\rho_.(\xi, T) \text{ for each real } \lambda \geq 0, T \in [0, T'] \text{ and } \xi \in L^2(\mathcal{F}_T).$$

Note that this property is generally not required for dynamic risk measures.

From now on, we assume that the driver f satisfies Assumption 4.1 with $T = T'$. This leads to the following additional properties for the risk measure ρ:

• **Monotonicity.** ρ is nonincreasing with respect to ξ, that is : for each $T \in [0, T']$, and $\xi^1, \xi^2 \in L^2(\mathcal{F}_T)$

(MO) If $\xi^1 \geq \xi^2$ a.s. , then $\rho_t(\xi^1, T) \leq \rho_t(\xi^2, T)$, $0 \leq t \leq T$ a.s.

Indeed, by the comparison theorem for BSDEs with jumps (see Theorem 4.2) applied to $f_1 = f_2 = f$, we get $X_t(\xi^1, T) \geq X_t(\xi^2, T)$, $0 \leq t \leq T$ a.s.

• **Convexity.** If f is concave with respect to (x, π, l), then the dynamic risk-measure ρ is *convex*, that is, for any $\lambda \in [0,1], T \in [0, T'], \xi^1, \xi^2 \in L^2(\mathcal{F}_T)$,

$$\rho(\lambda \xi^1 + (1-\lambda)\xi^2, T) \leq \lambda \rho(\xi^1, T) + (1-\lambda)\rho(\xi^2, T). \tag{5.40}$$

The proof is an application of the comparison theorem and is given in Appendix A.2.3.

Suppose moreover that in Assumption 4.1, we have $\theta_t^{x,\pi,l_1,l_2} > -1$.
By the strict comparison theorem (see Corollary 4.4), we derive the following property.

• **No Arbitrage.** The dynamic risk measure ρ satisfies the *no arbitrage* property :
for each $T \in [0, T']$, and $\xi^1, \xi^2 \in L^2(\mathcal{F}_T)$
(NA) If $\xi^1 \geq \xi^2$ a.s. and if $\rho_t(\xi^1, T) = \rho_t(\xi^2, T)$ a.s. on an event $A \in \mathcal{F}_t$, then $\xi^1 = \xi^2$ a.s. on A.

Note that generally, contrary to the monotonicity property, the *no arbitrage* property is not required for risk-measures.

The inverse problem. We now look at the inverse problem: when can a dynamic risk-measure be represented by a BSDE with jumps? The following proposition gives the answer.

Proposition 5.2 *Suppose that the intensity measure ν of the Poisson random measure satisfies $\int_{\mathbf{R}^*}(1 \wedge u^2)\nu(du) < +\infty$. Let ρ be a dynamic risk measure, that is, a map which, to each $\xi \in L^2(\mathcal{F}_T)$ and $T \geq 0$, associates an adapted RCLL process $(\rho_t(\xi, T))_{\{0 \leq t \leq T\}}$. Suppose that ρ is nonincreasing, consistent, translation invariant and satisfies the zero-one law as well as the no arbitrage property. Moreover suppose that there exists $C > 0$ and $-1 < C_1 \leq 0$ such that*

$$\rho_t(\xi + \xi', T) - \rho_t(\xi, T) \geq -X_t^{C,C_1}(\xi', T), \tag{5.41}$$

for any $\xi, \xi' \in L^2(\mathcal{F}_T)$, where $X_t^{C,C_1}(\xi', T)$ is the solution of the BSDE associated to terminal condition ξ' and driver $f_{C,C_1}(t, \pi, \ell) := C|\pi| + C\int_{\mathbf{R}^*}(1 \wedge |u|)\ell^+(u)\nu(du) - C_1\int_{\mathbf{R}^*}(1 \wedge |u|)\ell^-(u)\nu(du)$.

Then, there exists a Lipschitz driver $f(t, \pi, \ell)$ such that $\rho = \rho^f$, that is, ρ is the dynamic risk measure induced by a BSDE with jumps with driver $f(t, \pi, \ell)$.

This proposition corresponds to Theorem 4.6 in [25], here written in terms of risk measures. For the proof, we refer to [25].

5.2 Representation of convex dynamic risk measures

We now provide a dual representation for dynamic risk measures induced by concave BSDEs with jumps (which thus are convex risk measures).
Let f be a given driver independent of x. For each (ω, t), let $F(\omega, t, \cdot, \cdot, \cdot)$ be the polar function of f with respect to (π, ℓ), defined for each (α^1, α^2) in $\mathbf{R} \times L^2_\nu$ by

$$F(\omega, t, \alpha^1, \alpha^2) := \sup_{(\pi,\ell)\in\mathbf{R}^2\times L^2_\nu} [f(\omega, t, \pi, \ell) - \alpha^1 \pi - \langle \alpha^2, \ell \rangle_\nu]. \tag{5.42}$$

Theorem 5.1 *Suppose that the Hilbert space L^2_ν is separable. Let f be a Lipschitz driver with Lipschitz constant C, which does not depend on x. Suppose also that f satisfies Assumption 4.1, is continuous and concave with respect to (π, ℓ).*

Let $T \in [0, T']$. Let $\mathcal{A}_T$ be the set of predictable processes $\alpha = (\alpha^1, \alpha^2(.))$ such that $F(t, \alpha^1_t, \alpha^2_t)$ belongs to $I\!H^2_T$, where F is defined by (5.42). For each $\alpha \in \mathcal{A}_T$, let Q^α be the probability absolutely continuous with respect to P which admits Z^α_T as density with respect to P on $\mathcal{F}_T$, where Z^α is the solution of

$$dZ^\alpha_t = Z^\alpha_{t^-} \left(\alpha^1_t dW_t + \int_{\mathbf{R}^*} \alpha^2_t(u) d\tilde{N}(dt, du) \right) ; \quad Z^\alpha_0 = 1. \tag{5.43}$$

The convex risk-measure $\rho(., T)$ has the following representation: for each $\xi \in L^2(\mathcal{F}_T)$,

$$\rho_0(\xi, T) = \sup_{\alpha\in\mathcal{A}_T} [E_{Q^\alpha}(-\xi) - \zeta(\alpha, T)], \tag{5.44}$$

where for each $\alpha \in \mathcal{A}_T$,

$$\zeta(\alpha, T) := E_{Q^\alpha} \left[\int_0^T F(s, \alpha^1_s, \alpha^2_s) ds \right]. \tag{5.45}$$

Moreover, for each $\xi \in L^2(\mathcal{F}_T)$, there exists $\bar{\alpha} = (\bar{\alpha}^1, \bar{\alpha}^2) \in \mathcal{A}_T$ such that

$$F(t, \bar{\alpha}^1_t, \bar{\alpha}^2_t) = f(t, \pi_t, l_t) - \bar{\alpha}^1_t \pi_t - \langle \bar{\alpha}^2_t, l_t \rangle_\nu, \quad 0 \le t \le T, \quad dP \otimes dt - \text{a.s.},$$

where (X, π, l) is the solution to the BSDE with driver f, terminal time T and terminal condition ξ. Also, the process $\bar{\alpha}$ is optimal for (5.44).

Remark 5.6 *The function ζ can be interpreted as a penalty function. Note that this representation is related to that obtained by Föllmer and Schied [13]. Here, the penalty function ζ is given in terms of the polar function F of the driver.*

Proof. Since f is continuous with respect to (π, ℓ) and since $\mathbf{R} \times L^2_\nu$ admits a dense countable subset I, the supremum in (5.42) coincides with the supremum over I which implies the measurability of F. By results of convex analysis in Hilbert spaces (see e.g. Ekeland and Temam (1976) [8]), the polar function F is convex and lower semicontinuous with respect to α^1, α^2 since it is the supremum of continuous and convex functions.

Also, since f is concave and continuous, f and F satisfy the conjugacy relation, that is,

$$f(\omega, t, \pi, \ell) = \inf_{\alpha \in D_t(\omega)} \{F(\omega, t, \alpha_1, \alpha_2) + \alpha_1 \pi + \langle \alpha_2 , \ell \rangle_\nu\},$$

where for each (t, ω), $D_t(\omega)$ is the non empty set of $\alpha = (\alpha_1, \alpha_2) \in \mathbf{R} \times L^2_\nu$ such that $F(\omega, t, \alpha_1, \alpha_2) > -\infty$. Now, the following lemma holds.

Lemma 5.1 *For each (t, ω), $D_t(\omega) \subset U$, where U is the subset of the Hilbert space $\mathbf{R} \times L^2_\nu$ of the elements $\alpha = (\alpha_1, \alpha_2)$ such that α_1 is bounded by C and $\nu(du) -$ a.s*

$$\alpha_2(u) \geq -1 \quad and \quad |\alpha_2(u)| \leq \psi(u) \wedge C, \tag{5.46}$$

where C is the Lipschitz constant of f.

For each process $\alpha_t = (\alpha^1_t, \alpha^2_t) \in \mathcal{A}_T$, let f^α be the associated linear driver defined by

$$f^\alpha(\omega, t, \pi, \ell) := F(\omega, t, \alpha^1_t(\omega), \alpha^2_t(\omega)) + \alpha^1_t(\omega) \pi + \langle \alpha^2_t(\omega) , \ell \rangle_\nu.$$

Note first that for each $\alpha \in \mathcal{A}_T$, $f^\alpha \geq f$.

Let $T \in [0, T']$ and $\xi \in L^2(\mathcal{F}_T)$. Let $(X(\xi, T), \pi(\xi, T), l(\xi, T))$(also denoted (X, π, l)) be the solution of the BSDE associated with driver f, terminal time T and terminal condition ξ. The following lemma holds.

Lemma 5.2 *There exists a process $\bar{\alpha} = (\bar{\alpha}^1, \bar{\alpha}^2(.)) \in \mathcal{A}_T$ such that*

$$f(t, \pi_t, \ell_t) = \text{ess} \inf_{\alpha_t \in \mathcal{A}_T} \{f^\alpha(t, \pi_t, \ell_t)\} = f^{\bar{\alpha}}(t, \pi_t, \ell_t), \ 0 \leq t \leq T, \quad dP \otimes dt - \text{a.s.}$$

By the optimization principle for BSDEs with jumps (see Proposition 4.5), we thus derive that

$$X_0(\xi, T) = \inf_{\alpha \in \mathcal{A}_T} X^\alpha_0(\xi, T) = X^{\bar{\alpha}}_0(\xi, T) \tag{5.47}$$

where for each process $\alpha \in \mathcal{A}_T$, $X^\alpha(\xi, T)$ is the solution of the linear BSDE associated with driver f^α, terminal time T and terminal condition ξ. Let

$\alpha = (\alpha^1, \alpha^2) \in \mathcal{A}_T$. By Lemma 5.1, $|\alpha_t^2(u)| \le \psi(u) \wedge C \; dP \otimes dt \otimes \nu(du)$ a.s. Hence, by Proposition 3.2, the process Z^α, defined by (5.43), belongs to S^2. Consequently, by the representation formula of linear BSDEs (see (3.8)), we have

$$X_0^\alpha(\xi, T) = E\left[Z_T^\alpha \xi + \int_0^T Z_s^\alpha F(s, \alpha_s^1, \alpha_s^2) ds \right].$$

Now, by Lemma 5.1, we also have that $\alpha_t^2 \ge -1 \; dt \otimes dP \otimes d\nu$-a.s. Hence, $(Z_t^\alpha)_{0 \le t \le T}$ is a non negative martingale and the probability Q^α which admits Z_T^α as density with respect to P on $\mathcal{F}_T$ is well defined. We thus have

$$X_0^\alpha(\xi, T) = E_{Q^\alpha}\left[\xi + \int_0^T F(s, \alpha_s^1, \alpha_s^2) ds \right],$$

which completes the proof of the theorem. □

Proof of Lemma 5.1 Without loss of generality, we can suppose that Assumption 4.1 is satisfied for each (ω, t). Let $(t, \omega) \in [0, T] \times \Omega$ and let $\alpha = (\alpha_1, \alpha_2) \in D_t(\omega)$.
Let us first show that $\alpha_2 \ge -1$ ν-a.s. Suppose by contradiction that

$$\nu(\{u \in \mathbb{R}^*, \alpha_2(u) < -1\}) > 0.$$

Since f satisfies Assumption 4.1, the following inequality

$$f(\omega, t, 0, l) \ge f(\omega, t, 0, 0) + \langle \theta_t^{0,l,0}(\omega) \, , \, l \rangle_\nu.$$

holds for each $l \in L_\nu^2$. This together with the definition of the polar function F (see (5.42)) yields that

$$\begin{aligned} F(\omega, t, \alpha_1, \alpha_2) &\ge f(\omega, t, 0, l) - \langle \alpha_2 \, , \, l \rangle_\nu \\ &\ge f(\omega, t, 0, 0) + \langle \theta_t^{0,l,0}(\omega) - \alpha_2 \, , \, l \rangle_\nu. \end{aligned}$$

By applying this inequality to $l := n \mathbf{1}_{\{\alpha_2 < -1\}}$, where $n \in I\!N$, we thus derive that,

$$F(\omega, t, \alpha_1, \alpha_2) \ge f(\omega, t, 0, 0) + n \int_{\{\alpha_2 < -1\}} \left(\theta_t^{0,l,0}(\omega, u) - \alpha_2(u) \right) \nu(du),$$

and this holds for each $n \in I\!N$. Now, $\theta_t^{0,l,0}(\omega, u) \ge -1$. By letting n tend to $+\infty$ in this inequality, we get $F(\omega, t, \alpha_1, \alpha_2) = +\infty$, which provides the expected contradiction since $(\alpha_1, \alpha_2) \in D_t(\omega)$. We thus have proven that $\alpha_2 \ge -1$ ν-a.s.

By similar arguments, one can show that α_1 is bounded by C and that $|\alpha_2(u)| \leq \psi(u) \wedge C\ \nu(du)$-a.s. , which ends the proof. □

Proof of Lemma 5.2 The proof is divided in two steps.
Step 1: Let us first prove that for each (ω, t), there exists $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2) \in U$ such that

$$\inf_{\alpha \in U}\{F(\omega,t,\alpha_1,\alpha_2)+\alpha_1\,\pi_t(\omega)+\langle \alpha_2\,, l_t(\omega)\rangle_\nu\} = F(\omega,t,\bar{\alpha}_1,\bar{\alpha}_2)+\bar{\alpha}_1\,\pi_t(\omega)+\langle \bar{\alpha}_2\,, l_t(\omega)\rangle_\nu. \tag{5.48}$$

The proof is based on classical arguments of convex analysis. Fix (ω, t). The set U is strongly closed and convex in $\mathbf{R} \times L^2_\nu$. Hence, U is closed for the weak topology. Moreover, since U is bounded, it is a compact set for the weak topology. Let ϕ be the function defined for each $\alpha = (\alpha_1, \alpha_2) \in \mathbf{R} \times L^2_\nu$ by

$$\phi(\alpha) = F(\omega,t,\alpha_1,\alpha_2) + \alpha_1\,\pi_t(\omega) + \langle \alpha_2\,, l_t(\omega)\rangle_\nu.$$

This function is convex and lower semi-continuous (l.s.c.) for the strong topology in $\mathbf{R} \times L^2_\nu$. By classical results of convex analysis, it is l.s.c. for the weak topology. Now, there exists a sequence $\alpha^n = (\alpha^n_1, \alpha^n_2)_{n\in\mathbb{N}}$ of U such that $\phi(\alpha^n) \to \inf_{\alpha\in U}\phi(\alpha)$ as $n \to \infty$.
Since U is weakly compact, there exists an extracted sequence still denoted by (α^n) which converges for the weak topology to $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2)$ for some $\alpha \in U$. Since ϕ is l.s.c. for the weak topology, it follows that

$$\phi(\bar{\alpha}) \leq \liminf \phi(\alpha^n) \;\; = \;\; \inf_{\alpha\in U}\phi(\alpha)\,.$$

Therefore, $\phi(\bar{\alpha}) = \inf_{\alpha\in U}\phi(\alpha)$. Hence $\bar{\alpha}$ satisfies (5.48), which ends the proof of step 1.
Step 2: Let us now introduce the set $\mathcal{U}$ of processes α: $[0,T]\times\Omega \to \mathbf{R}\times L^2_\nu$; $(t,\omega) \mapsto \alpha_t(\omega,.)$ which are measurable with respect to σ-algebras $\mathcal{P}$ and $\mathcal{B}(\mathbf{R})\otimes\mathcal{B}(L^2_\nu)$ and which take their values in U $dP\otimes dt$-a.s.

Since the Hilbert space L^2_ν is supposed to be separable, it is a polish space. Hence, the section theorem §81 in the Appendix of Ch.III in Dellacherie and Meyer (1975) can be applied. It follows that there exists a process $\bar{\alpha} = (\bar{\alpha}^1, \bar{\alpha}^2(.))$ which belongs to $\mathcal{U}$ such that, $dP\otimes dt$-a.s. ,

$$f(t,\pi_t,\ell_t) = \operatorname{ess}\inf_{\alpha_t\in\mathcal{U}}\{f^\alpha(t,\pi_t,\ell_t)\} = f^{\bar{\alpha}}(t,\pi_t,\ell_t),\ 0 \leq t \leq T \quad dP\otimes dt - \text{a.s.} \tag{5.49}$$

Let us show that the process $\bar{\alpha}^2_t(.)$ is *predictable*. Since L^2_ν is a separable Hilbert space, there exists a countable orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of L^2_ν. For each $i \in \mathbb{N}$, defined $\lambda^i_t(\omega) = \langle \bar{\alpha}^2_t(\omega), e_i\rangle_\nu$. Since the map $\langle\,.\,, e_i\rangle_\nu$ is continuous on L^2_ν, the process (λ^i_t) is $\mathcal{P}$-measurable. As $\bar{\alpha}^2_t(u) = \sum_i \lambda^i_t(\omega)e_i(u)$, it follows that $\bar{\alpha}^2$: $[0,T] \times \Omega \times \mathbf{R}^* \to \mathbf{R}$; $(t,\omega,u) \mapsto \bar{\alpha}_t^2(\omega,u)$ is $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^*)$-measurable. It is thus *predictable* . Equality (5.49) together with the definition of $f^{\bar{\alpha}}$ yields that

$$F(t,\bar{\alpha}_t^1,\bar{\alpha}_t^2) = f(t,\pi_t,l_t) - \bar{\alpha}_t^1 \, \pi_t - \langle \bar{\alpha}_t^2 \, , l_t \rangle_\nu \ \ 0 \le t \le T, \ \ dP \otimes dt - \text{a.s.} \, ,$$

which implies that the process $F(t,\bar{\alpha}_t^1,\bar{\alpha}_t^2)$ belongs to $I\!H_T^2$ as a sum of processes in $I\!H_T^2$. Hence, $(\bar{\alpha}_t) \in \mathcal{A}_T$, which ensures that equality (5.49) still holds with $\mathcal{U}$ replaced by $\mathcal{A}_T$. □

If f is positively homogeneous, then $F = 0$. Consequently the penalty function ζ is equal to zero, which corresponds to the result of Föllmer and Schied [13]. Moreover, the set $\mathcal{A}_T$ is equal for all T to the set of predictable processes (α_t) valued in D_t $dP \otimes dt$-a.s.

Remark 5.7 *Due to the presence of jumps, the proof requires fine mathematical results since the controls α are here not valued in a finite dimensional space as $\mathbf{R}^p$, but in a Hilbert space, more precisely, $\mathbf{R} \times L_\nu^2$.*

Note that the separability assumption made on the Hilbert space L_ν^2 is used in the proof to solve some measurability problems. In particular, it allows us to apply the section theorem of [6]. Indeed, this theorem requires that the space is lusinian, that is, isomorph to a borelian part of a polish space.

In the dual representation, the supremum cannot be generally taken over the probability measures Q^α equivalent to P. More precisely, let $\alpha^2 \in D_t$, and hence, such that $\alpha^2 \ge -1$. The point is that F is l.s.c. but not necessarily continuous, and the domain D_t of F is not necessarily an open set. It is possible that there does not exist any sequence (α_n^2) in D_t converging to α^2 with $\alpha_n^2 > -1$.

In the Brownian case, the probability measures Q^α are equivalent to P, and the above dual representation corresponds to that provided by El Karoui et al. (1997) for concave BSDEs.

5.3 Dynamic risk-measures under model ambiguity

We consider now dynamic risk-measures in the case of model ambiguity, parameterized by a control α as follows. Let A be a polish space and let $\mathcal{A}$ the set of A-valued predictable processes α. To each coefficient $\alpha \in \mathcal{A}$, is associated a model via a probability measure Q^α called *prior* as well as a dynamic risk measure ρ^α. More precisely, for each $\alpha \in \mathcal{A}$, let Z^α be the solution of the SDE:

$$dZ_t^\alpha = Z_{t^-}^\alpha \left(\beta^1(t,\alpha_t) dW_t + \int_{\mathbf{R}^*} \beta^2(t,\alpha_t,u) d\tilde{N}(dt,du) \right) \; ; \quad Z_0^\alpha = 1,$$

where $\beta^1 : (t, \omega, \alpha) \mapsto \beta^1(t, \omega, \alpha)$, is a $\mathcal{P} \otimes \mathcal{B}(A)$-measurable function defined on $[0, T'] \times \Omega \times A$ and valued in $[-C, C]$, with $C > 0$, and $\beta^2 : (t, \omega, \alpha, u) \mapsto \beta^2(t, \omega, \alpha, u)$ is a $\mathcal{P} \otimes \mathcal{B}(A) \otimes \mathcal{B}(\mathbf{R}^*)$-measurable function defined on $[0, T'] \times \Omega \times A \times \mathbf{R}^*$ which satisfies $dt \otimes dP \otimes d\nu(u)$-a.s.

$$\beta^2(t, \alpha, u) \geq C_1 \quad \text{and} \quad |\beta^2(t, \alpha, u)| \leq \psi(u), \tag{5.50}$$

with $C_1 > -1$ and ψ is a bounded function $\in L^p_\nu$ for all $p \geq 1$. Hence, $Z^\alpha_{T'} > 0$ a.s. and, by Proposition A.1, $Z^\alpha_{T'} \in L^p(\mathcal{F}_{T'})$ for all $p \geq 1$.

For each $\alpha \in \mathcal{A}$, let Q^α be the probability measure equivalent to P which admits $Z^\alpha_{T'}$ as density with respect to P on $\mathcal{F}_{T'}$. By Girsanov's theorem (see [17]), the process $W^\alpha_t := W_t - \int_0^t \beta^1(s, \alpha_s)ds$ is a Brownian motion under Q^α and N is a Poisson random measure independant from W^α under Q^α with compensated process $\tilde{N}^\alpha(dt, du) = \tilde{N}(dt, du) - \beta^2(t, \alpha_t, u)\nu(du)dt$. Even if the filtration $\mathcal{F}$ is not generated by W^α and $\tilde{N}^\alpha$, we have a representation theorem for Q^α-martingales with respect to W^α and $\tilde{N}^\alpha$. More precisely,

Lemma 5.2 *Let (M_t) be a local martingale under Q^α, locally p-integrable under Q^α, for some $p > 2$. Then, there exists a unique pair of predictable processes $(\pi_t, l_t(.))$ such that*

$$M_t = M_0 + \int_0^t \pi_s dW^\alpha_s + \int_0^t \int_{\mathbf{R}^*} l_s(u)\tilde{N}^\alpha(ds, du) \quad 0 \leq t \leq T \quad \text{a.s.} \tag{5.51}$$

Proof. Suppose first that (M_t) is p-integrable under Q^α. Since (M_t) is a Q^α-martingale, the process $N_t := Z^\alpha_t M_t$ is a martingale under P. By Assumption (5.50), it follows from Proposition A.1 that $Z^\alpha_{T'} \in L^q(\mathcal{F}_{T'})$ for all $q \geq 1$. By Hölder's inequality, N_T is thus square integrable under P. Consequently, we can apply the martingale representation theorem of Tang and Li [26] of P-locally square integrable martingales with respect to W and $\tilde{N}$. Thus, there exists a unique pair of predictable processes $(\psi_t, k_t(.))$ such that

$$N_t = N_0 + \int_0^t \psi_s dW_s + \int_0^t \int_{\mathbf{R}^*} k_s(u)\tilde{N}(ds, du) \quad 0 \leq t \leq T \quad \text{a.s.}$$

By applying Itô's formula to $M_t = N_t(Z^\alpha_t)^{-1}$ and by classical computations, one can derive the existence of $(\pi_t, l_t(.))$ satisfying (5.51). The case of locally p-integrable martingales under Q^α follows from localization arguments. $\square$

For each control α, the associated dynamic risk measure will be induced by a BSDE under Q^α and driven by W^α and $\tilde{N}^\alpha$, which makes sense since we have the above Q^α-martingale representation property. Let us first introduce

a function
$F : [0,T'] \times \Omega \times \mathbf{R} \times L^2_\nu \times A \to \mathbf{R}$; $(t, \omega, \pi, \ell, \alpha) \mapsto F(t, \omega, \pi, \ell, \alpha)$ which is $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(L^2_\nu) \otimes \mathcal{B}(A)$-measurable. Suppose F is uniformly Lipschitz with respect to (π, ℓ), continuous with respect to α, and such that $\operatorname{ess\,sup}_{\alpha \in A} |F(\cdot, t, 0, 0, 0, \alpha)| \in I\!H^{p,T}$, for each $p \geq 2$. Suppose also that

$$F(t, \pi, l_1, \alpha) - F(t, \pi, l_2, \alpha) \geq \langle \theta_t^{\pi, l_1, l_2, \alpha}\,,\, l_1 - l_2 \rangle_\nu, \tag{5.52}$$

for some adapted process $\theta_t^{\pi, l_1, l_2, \alpha}(\cdot)$ with

$$\theta : \Omega \times [0,T'] \times \mathbb{R} \times (\mathcal{L}^2_\nu)^2 \times A \mapsto \mathcal{L}^2_\nu$$

$\mathcal{P} \otimes \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathcal{L}^2_\nu)^2 \otimes \mathcal{B}(A)$-measurable and satisfying $|\theta_t^{\pi, l_1, l_2, \alpha}(u)| \leq \bar{\psi}(u)$, where $\bar{\psi}$ is bounded and in L^p_ν, for all $p \geq 1$, and $\theta_t^{\pi, l_1, l_2, \alpha} \geq -1 - C_1$.

For each $\alpha \in \mathcal{A}$, the associated driver is given by

$$F(t, \omega, \pi, \ell, \alpha_t(\omega)). \tag{5.53}$$

Note that these drivers are equi-Lipschitz. For each $\alpha \in \mathcal{A}$, let ρ^α be the dynamic risk-measure induced by the BSDE associated with driver $F(., \alpha_t)$ and driven by W^α and $\tilde{N}^\alpha$.
More precisely, fix $T \in [0, T']$ and $\xi \in L^p(\mathcal{F}_T)$ with $p > 2$. By Proposition A.1, $Z_T^\alpha \in L^q(\mathcal{F}_{T'})$ for all $q \geq 1$. Hence, by Hölder's inequality, $\xi \in L^2_\alpha$, where L^2_α denotes the space of random variables which are square integrable under Q^α. Similarly, $|F(\cdot, t, 0, 0, 0, \alpha_t)| \in I\!H^{2,T}_\alpha$. Hence, there exists a unique solution $(X^\alpha, \pi^\alpha, l^\alpha)$ in $S^2_\alpha \times I\!H^2_\alpha \times I\!H^2_{\alpha,\nu}$ of the Q^α-BSDE

$$-dX_t^\alpha = F(t, \pi_t^\alpha, l_t^\alpha, \alpha_t)dt - \pi_t^\alpha dW_t^\alpha - \int_{\mathbf{R}^*} l_t^\alpha(u) \tilde{N}^\alpha(dt, du); \qquad X_T^\alpha = \xi, \tag{5.54}$$

driven by W^α and $\tilde{N}^\alpha$. The dynamic risk-measure $\rho^\alpha(\xi, T)$ of position ξ is thus well defined by

$$\rho_t^\alpha(\xi, T) := -X_t^\alpha(\xi, T), \;\; 0 \leq t \leq T, \tag{5.55}$$

with $X^\alpha(\xi, T) = X^\alpha$. Assumption (5.52) yields the monotonicity property of ρ^α.

The agent is supposed to be averse to ambiguity. Her risk measure at time t is thus given, for each $T \in [0, T']$ and $\xi \in L^p(\mathcal{F}_T)$, $p > 2$, by

$$\operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} \rho_t^\alpha(\xi, T) = -\operatorname*{ess\,inf}_{\alpha \in \mathcal{A}} X_t^\alpha(\xi, T). \tag{5.56}$$

Note that it defines a monotonous dynamic risk measure.

We now show that this dynamic risk measure is induced by a BSDE driven by W and $\tilde{N}$ under probability P. More precisely,

Theorem 5.1 *Let f be the function defined for each (t,ω,π,ℓ) by*

$$f(t,\omega,\pi,\ell) := \inf_{\alpha\in A}\{F(t,\omega,\pi,\ell,\alpha) + \beta^1(t,\omega,\alpha)\pi + \langle\beta^2(t,\omega,\alpha),\ell\rangle_\nu\}. \quad (5.57)$$

Let ρ be the dynamic risk measure associated with driver f, defined for each $T\in[0,T']$ and $\xi\in L^p(\mathcal{F}_T)$ $(p>2)$, by

$$\rho_t(\xi,T) := -X_t(\xi,T), \;\; 0\le t\le T, \quad (5.58)$$

with $X(\xi,T)=X$, where (X,π,l) is the unique solution in $S^{2,T}\times I\!H^{2,T}\times I\!H^{2,T}_\nu$ of the P-BSDE associated with driver f, that is,

$$-dX_t = f(t,\pi_t,l_t)dt - \pi_t dW_t - \int_{\mathbf{R}^*} l_t(u)\tilde{N}(dt,du); \qquad X_T=\xi. \quad (5.59)$$

For each $T\in[0,T']$ and $\xi\in L^p(\mathcal{F}_T)$ with $p>2$, we have for each $t\in[0,T]$,

$$\rho_t(\xi,T) = \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}} \rho_t^\alpha(\xi,T) \quad \text{a.s.} \quad (5.60)$$

Proof. In order to prove this result, we will express the problem in terms of BSDEs under probability P and then apply the second optimization principle.

Fix now $\xi\in L^p(\mathcal{F}_T)$ with $p>2$. Since $(X^\alpha,\pi^\alpha,l^\alpha)$ is solution of BSDE (5.54), it clearly satisfies

$$-dX^\alpha_t = f^\alpha(t,\pi^\alpha_t,l^\alpha_t)dt - \pi^\alpha_t dW_t - \int_{\mathbf{R}^*} l^\alpha_t(u)\tilde{N}(dt,du); \qquad X^\alpha_T=\xi, \quad (5.61)$$

which is a P-BSDE driven by W and $\tilde{N}$, and where the driver is given by

$$f^\alpha(t,\pi,\ell) := F(t,\pi,\ell,\alpha_t) + \beta^1(t,\alpha_t)\pi + \langle\beta^2(t,\alpha_t),\ell\rangle_\nu. \quad (5.62)$$

The drivers f^α are clearly equi-Lipschitz.
Let p' be a real number such that $2<p'<p$. Now, Z^α_T is q-integrable, for all $q\ge1$. Hence, by Hölder's inequality, $\xi\in L^{p'}_\alpha$, where $L^{p'}_\alpha$ denotes the space of random variables which are p'-integrable under Q^α. Similarly, $F(t,0,0,\alpha_t)\in I\!H^{p'}_\alpha$. By Proposition A.5 in the Appendix, there exists a unique solution $(X^\alpha,\pi^\alpha,l^\alpha)$ in $S^{p'}_\alpha\times I\!H^{p'}_\alpha\times I\!H^{p'}_{\alpha,\nu}$ of the Q^α-BSDE (5.54).

Now, suppose we have shown that $(Z^\alpha)^{-1}\in S^{q,T'}$ for all $q\ge1$. Since $p'>2$, by Hölder's inequality, we derive that $(X^\alpha,\pi^\alpha,l^\alpha)$ belongs to $S^2\times I\!H^2\times I\!H^2_\nu$ and is thus the unique solution of P-BSDE (5.61) in $S^2\times I\!H^2\times I\!H^2_\nu$. Moreover, for each α, f^α satisfies Assumption 4.1. Indeed, we have

$$\begin{aligned} f^\alpha(t,\pi,\ell_1) - f^\alpha(t,\pi,\ell_2) &= F(t,\pi,\ell_1,\alpha_t) - F(t,\pi,\ell_2,\alpha_t) + \langle\beta^2(t,\alpha_t),\ell_1-\ell_2\rangle_\nu \\ &\ge \langle\theta_t^{\pi,\ell_1,\ell_2} + \beta^2(t,\alpha_t)\,,\,\ell_1-\ell_2\rangle_\nu, \end{aligned}$$

with $\theta_t^{\pi,\ell_1,\ell_2}+\beta^2(t,\alpha_t) \geq (-1-C_1)+C_1 \geq -1$ and $|\theta_t^{\pi,\ell_1,\ell_2}+\beta^2(t,\alpha_t)| \leq \psi+\bar{\psi}$.

Let us show that f, defined by (5.57), is a Lipschitz driver. Since A is a polish space, there exists a countable subset D of A which is dense in A. As F is continuous with respect to α, it follows that the above equality still holds with A replaced by D, which gives that f is $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(L^2_\nu)$-measurable. Also, f is Lipschitz and $f(\cdot, 0, 0) \in I\!H^{2,T}$, which yields that f is a Lipschitz driver.

By the definitions of f (see (5.57)) and f^α (see (5.62)), we get that for each $\alpha \in \mathcal{A}$, $f \leq f^\alpha$.

Also, for each $\epsilon > 0$ and each $(t, \omega, \pi, l) \in \Omega \times [0, T'] \times \mathbf{R} \times L^2_\nu$, there exists $\alpha^\epsilon \in A$ such that

$$f(t,\omega,\pi,\ell)+\epsilon \geq F(t,\omega,\pi,\ell,\alpha^\epsilon)+\beta^1(t,\omega,\alpha^\epsilon)\pi+\langle \beta^2(t,\omega,\alpha^\epsilon),\ell\rangle_\nu.$$

By the section theorem of [6], for each $\epsilon > 0$, there exists an A-valued predictable process (α_t^ϵ) such that

$$f(t,\pi_t,l_t)+\epsilon \geq f^{\alpha^\epsilon}(t,\pi_t,l_t),\ 0 \leq t \leq T \ \ \mathrm{dP} \otimes \mathrm{dt} - a.s.$$

Consequently, by the second optimization principle for BSDEs (Theorem 4.6), equality (5.60) holds, which is the desired result.

It remains to show that $(Z^\alpha)^{-1} \in S_\alpha^{q,T'}$ for all $q > 1$. Now, by classical computations, we derive that $(Z^\alpha)^{-1}$ satisfies the following SDE:

$$d(Z_t^\alpha)^{-1} = (Z_{t^-}^\alpha)^{-1}\left(-\beta^1(t,\alpha_t)dW_t^\alpha - \int_{\mathbf{R}^*} \beta^2(t,\alpha_t,u)d\tilde{N}^\alpha(dt,du)\right)\ ;\quad (Z_0^\alpha)^{-1}=1.$$

Since β^1 is bounded and since β^2 satisfies (5.50), it follows from Proposition A.1 that $(Z^\alpha)_{T'}^{-1} \in L_\alpha^q$ for all $q > 1$. Hence, by martingale inequalities, $(Z^\alpha)^{-1} \in S_\alpha^{q,T'}$ for all $q > 1$. The proof is thus complete. □

Moreover, from Proposition 4.1, the following property follows.

Proposition 5.3 *Let (X, π, l) be the solution of the BSDE associated with driver f and terminal condition (T, ξ). Let $\bar{\alpha} \in \mathcal{A}$ and let $t \in [0, T]$.*

- *(Sufficient condition of optimality) Suppose that*

  $$f(s,\pi_s,l_s) = F(s,\pi_s,l_s,\bar{\alpha}_s)+\beta^1(s,\bar{\alpha}_s)\pi_s+\langle \beta^2(s,\bar{\alpha}_s),l_s\rangle_\nu,\quad t \leq s \leq T,\quad dP\otimes ds\text{–a.s.} \tag{5.63}$$

  Then, $\bar{\alpha}$ is t-optimal, that is, optimal for (5.60).

- *(Necessary condition of optimality) Suppose that $\bar{\alpha}$ is t-optimal and that $\beta^2(t,\bar{\alpha}_t)+\theta_t^{\pi_t,l_t^{\bar{\alpha}},l_t} > -1$, $dP\otimes dt$-a.s. (which is satisfied if, for example, $C_1 + C_1' > -1$).*
  Then equality (5.63) holds.

Remark 5.3 *Suppose A is compact and that F, β^1 and β^2 are continuous with respect to α. Then, for each $(t, \omega, \pi, l) \in \Omega \times [0, T'] \times \mathbf{R} \times L^2_\nu$, there exists $\bar{\alpha} \in A$ such that*

$$f(t, \omega, \pi, \ell) = F(t, \omega, \pi, \ell, \bar{\alpha}) + \beta^1(t, \bar{\alpha})\pi + \langle \beta^2(t, \bar{\alpha}), l \rangle_\nu.$$

By the section theorem of [6], there exists an A-valued predictable process $(\bar{\alpha}_t)$ such that (5.63) is satisfied with $t = 0$. It follows that $\bar{\alpha}$ is optimal for (5.60).

Example. We suppose that L^2_ν is separable and that A is a subset of the Hilbert space $\mathbf{R} \times L^2_\nu$ such that $A \subset [-K, K] \times \Upsilon$, where

$$\Upsilon := \{\varphi \in \mathcal{P},\ \ C'_1 \leq \varphi(u) \ \text{ and } |\varphi(u)| \leq \psi(u) \ \ \nu(du) \text{ a.s. } \},$$

with $C'_1 > -1$ and ψ is bounded and in L^p_ν, for all $p \geq 1$. In this case, for each process $\alpha := (\alpha^1, \alpha^2) \in \mathcal{A}$, the prior Q^α admits $Z^\alpha_{T'}$ as density with respect to P, Z^α being the solution of the (5.43). Theorem 5.1 can then be applied.

Remark 5.8 *In the case of a controlled linear driver, that is, when $F(t, \omega, \pi, \ell, \alpha_t(\omega))$ is linear with respect to π and ℓ, the above problem is related to some classical control problems (see for example [10] in the Brownian case).*

Note also that the above results still hold if ψ is only supposed to belong to L^3_ν. More precisely, by Proposition A.1, $Z^\alpha_{T'} \in L^3(\mathcal{F}_{T'})$. Then, equality (5.58) holds but only for $\xi \in L^6(\mathcal{F}_T)$.

5.4 Some links between dynamic risk-measures induced by BSDEs and the instantaneous interest rate

In this section, we explain why, from a financial point of view, the dependance of $f(t, x, \pi, l)$ with respect to x is relevant for dynamic risk measures modeling. More precisely, it allows us to take into account the instantaneous interest rate in the market or some ambiguity on this rate. To our knowledge, this point had not really been stressed in the literature on risk measures, even in the Brownian case.

5.4.1 Case where the driver f is linear with respect to x

Let f be a Lipschitz driver, linear with respect to x, that is, which can be written as

$$f(., t, x, \pi, l) = -r_t x + g(., t, \pi, l),$$

where g a Lipschitz driver which does not depend on x and where (r_t) is a bounded predictable process, which can be interpreted as an instantaneous interest rate. Let us denote by ρ the associated risk-measure. Let $T \in [0, T']$ and let $\xi \in L^2(\mathcal{F}_T)$. Set $X_t := X_t(\xi, T)$. Let us consider $\tilde{X}_t := e^{-\int_0^t r_s ds} X_t$, which can be seen as the discounted process. One can show that $\tilde{X}$ is the solution of a BSDE associated with driver $\tilde{g}(., t, \pi, l) = e^{-\int_0^t r_s ds} g(., t, e^{-\int_0^t r_s ds}\pi, e^{-\int_0^t r_s ds} l)$ and with terminal condition $e^{-\int_0^T r_s ds}\xi$. The risk measure ρ thus reduces to a new risk measure $\tilde{\rho}$ associated to driver $\tilde{g}$. More precisely, we have

$$e^{-\int_0^t r_s ds} \rho_t(\xi, T) = \tilde{\rho}_t(e^{-\int_0^T r_s ds}\xi, T), \quad 0 \leq t \leq T, \quad \text{a.s.}$$

for each $\xi \in L^2(\mathcal{F}_T)$. In particular, for initial time $t = 0$, we get $\rho_0(\xi, T) = \tilde{\rho}_0(e^{-\int_0^T r_s ds}\xi, T)$. This new measure $\tilde{\rho}$, which operates on discounted positions, is translation-invariant because $\tilde{g}$ does not depend on x. We thus have for each $L^2(\mathcal{F}_T)$-measurable variable ξ (position) and each constant $m \in \mathbf{R}$

$$\rho_0(\xi + m e^{\int_0^T r_s ds}, T) = \tilde{\rho}_0(e^{-\int_0^T r_s ds}\xi + m, T) = \tilde{\rho}_0(e^{-\int_0^T r_s ds}\xi, T) - m = \rho_0(\xi, T) - m.$$

In other words, the risk measure ρ is (r_t)-translation invariant (here at time 0), which makes sense from a financial point of view. More precisely, the constant m can be interpreted as an initial amount which is invested in the riskless asset with intantaneous interest rate (r_t). An analogous property can be similarly proven at any time $t \in [0, T]$.

5.4.2 Case of a concave driver f with respect to x

We now consider the case when the driver $f(t, x, \pi, l)$ is concave with respect to x (but not necessarily with respect to (x, π, l)) and show that it can be interpreted in terms of ambiguity on the instantaneous interest rate process. For each (ω, t, π, l), let $F(\omega, t, \cdot, \pi, l)$ be the polar function of f with respect to x, defined for each δ in $\mathbf{R}$ by

$$F(\omega, t, \delta, \pi, l) := \sup_{x \in \mathbf{R}} [f(\omega, t, x, \pi, \ell) - \delta\, x].$$

Proposition 5.4 *[Robust representation of ρ] Suppose that f is a Lipschitz driver with Lipschitz constant C, satisfying Assumption 4.1 and concave with respect to x.*
Let ρ be the dynamic risk-measure induced by the BSDE associated with driver f.
Let $T \in [0, T']$. Let $\mathcal{D}_T$ be the set of predictable processes δ valued in $\mathbf{R}$ such

that $F(t, \delta_t, 0, 0)$ belongs to $I\!H_T^2$. For each **R***-valued predictable process δ, let f^δ be the driver defined by*

$$f^\delta(\omega, t, x, \pi, \ell) := F(\omega, t, \delta_t(\omega), \pi, l) + \delta_t(\omega)\, x.$$

For each $\xi \in L^2(\mathcal{F}_T)$ and for each $t \in [0, T]$, we have

$$\rho_t(\xi, T) = \text{ess}\sup_{\delta \in \mathcal{D}_T} \rho_t^\delta(\xi, T) \quad a.s.,$$

where for each $\delta \in \mathcal{D}_T$, ρ^δ is the dynamic risk-measure induced by the non linear BSDE associated with driver f^δ.

The coefficients $(-\delta,\ \delta \in \mathcal{D}_T)$ can be interpreted as possible intantaneous interest rates when there is ambiguity on the interest rate. Also, as seen above, the risk-measure ρ^δ can be interpreted as a risk-measure in a market with interest rate process $(-\delta_t)$.

Proof. Since f is concave with respect to x, f and F satisfy the conjugacy relation

$$f(\omega, t, x, \pi, \ell) = \inf_{\delta \in D_t(\omega, \pi, l)} \{F(\omega, t, \delta, \pi, l) + \delta\, x\},$$

where for each (t, ω, π, l), $D_t(\omega, \pi, l)$ is the non empty set of reals δ such that $F(\omega, t, \delta, \pi, l) < +\infty$. Since f is Lipschitz with constant C, $D_t(\omega, \pi, l)$ is bounded by C.

Let $T > 0$ and $\xi \in L^2(\mathcal{F}_T)$. Let $(X, \pi, l))$ be the solution of the BSDE associated with f, ξ, T. For each (ω, t), since $F(\omega, t, ., \pi_t(\omega), l_t(\omega))$ is convex and hence continuous, there exists $\bar{\delta} \in [-C, C]$ such that

$$\inf_{\delta \in [-C, C]} \{F(\omega, t, \delta, \pi_t(\omega), l_t(\omega)) + \delta\, x\} = F(\omega, t, \bar{\delta}, \pi_t(\omega), l_t(\omega)) + \bar{\delta}\, x$$

Let $\mathcal{C}$ be the set of predictable processes (δ_t) bounded by C. By the section theorem of [6]), there exists a predictable process $\bar{\delta}_t \in \mathcal{C}$ such that

$$f(t, X_{t^-}, \pi_t, \ell_t) = \text{ess}\inf_{(\delta_t) \in \mathcal{C}} \{f^\delta(t, X_{t^-}, \pi_t, \ell_t)\} = f^{\bar{\delta}}(t, X_{t^-}, \pi_t, \ell_t),\ 0 \le t \le T\ dP \otimes dt - \text{a.s.}$$

It follows that the process $F(t, \bar{\delta}_t, \pi_t, l_t)$ belongs to $I\!H_T^2$ as a sum of processes in $I\!H_T^2$. Hence, $(\bar{\delta}_t) \in \mathcal{D}_T$, which ensures that the above equalities still holds with $\mathcal{C}$ replaced by $\mathcal{D}_T$. The result follows by applying Proposition 4.5. □

Remark 5.9 *In the particular case when the driver is concave with respect to all variables (x, π, ℓ), proceeding as in Subsection 5.2, we obtain the following representation :*

$$\rho_0(\xi, T) = \sup_{\alpha, \delta} J^{\alpha, \delta}(\xi, T). \tag{5.64}$$

where

$$J^{\alpha,\delta}(\xi, T) = E_{Q^\alpha}[-e^{\int_0^T \delta_u du}\xi] - E_{Q^\alpha}\left[\int_0^T e^{\int_0^s \delta_u du}\ F(s, \alpha_s^1, \alpha_s^2)ds\right].$$

However, since $J^{\alpha,\delta}$ is affine with respect to ξ, it is not relevant in this case to interpret it as a risk measure. In the Brownian case, equality (5.64) corresponds to that in [10] and [12].

5.4.3 Model with ambiguity on the interest rate and on the model

In this section, we consider the model described in Section 5.3 but when there is also ambiguity on the instantaneous interest rate process. To each control $\alpha \in \mathcal{A}$, corresponds an instantaneous interest rate process $\delta(t, \alpha_t)$, where $\delta : (t, \omega, \alpha) \mapsto \delta(t, \omega, \alpha)$ is a $\mathcal{P} \otimes \mathcal{B}(A)$-measurable function defined on $[0, T'] \times \Omega \times A$ and valued in $[-C, C]$, where $C > 0$. Also, for each $\alpha \in \mathcal{A}$, the associated driver is given here by

$$F(t, \pi, \ell, \alpha_t) + \delta(t, \alpha_t)\, x, \tag{5.65}$$

instead of (5.53). Also, ρ^α is the dynamic risk-measure induced by the Q^α-BSDE associated with driver (5.65) and driven by W^α and $\tilde{N}^\alpha$.

In this case, we have $\rho_t(\xi, T) = \operatorname{ess\,sup}_{\alpha \in \mathcal{A}} \rho_t^\alpha(\xi, T)$ a.s., where ρ is the dynamic risk-measure induced by the P-BSDE associated with driver f, given by

$$f(t, \omega, x, \pi, \ell) = \inf_{\alpha \in A}\{F(t, \omega, \pi, \ell, \alpha) + \delta(t, \omega, \alpha)x + \beta^1(t, \omega, \alpha)\pi + \langle \beta^2(t, \omega, \alpha), \ell\rangle_\nu\}.$$

The driver f thus depends on x (and is not necessarily concave w.r. to x).

A Appendix

A.1 Exponential local martingales

A.1.1 Classical properties of exponential local martingales

We recall some useful properties on exponential local martingales which can be found in [7], Chapter 8. We are given a filtered propability space which satisfies the usual conditions.

Let M and N be local martingales, locally square integrable. There exists a unique predictable process of finite variation denoted by $\langle M, N\rangle$ such that $MN - \langle M, N\rangle$ is a local martingale equal to 0 at time 0.

Let M be a local martingale with $M_0 = 0$. Then M admits the following unique decomposition $M = M^c + M^d$ where M^c is a continuous local martingale and M^d is a purely discontinuous local martingale given by $M_t^d = \sum_{0<s\le t} \Delta M_s$.

The quadratic variation of M is the process given by:

$$[M, M]_t = \langle M^c, M^c\rangle_t + \sum_{0<s\le t} \Delta M_s^2. \tag{A.66}$$

If X and Y are two semimartingales, $[X, Y]$ is defined by

$$[X, Y]_t := \langle M_X^c, M_Y^c\rangle_t + \sum_{0<s\le t} \Delta X_s \Delta Y_s, \tag{A.67}$$

where M_X^c (resp. M_Y^c) is the continuous part of the local martingale associated to X (resp. Y).

If X is a semimartingale with $X_0 = 0$, there exists a unique semimartingale Z denoted by $\mathcal{E}(X)$, which satisfies the equation: $Z_t = 1 + \int_0^t Z_{s^-} dX_s$ forall t. It is given by the so-called exponential formula of Doléans-Dade:

$$\mathcal{E}(X)_t := \exp\{X_t - \frac{1}{2}\langle M_X^c, M_X^c\rangle_t\} \prod_{r\le t}(1 + \Delta X_r)e^{-\Delta X_r} \tag{A.68}$$

We derive that, if Y is also a semimartingale with $Y_0 = 0$, then

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X, Y]). \tag{A.69}$$

Note that if $\Delta X \ge -1$ (resp. > -1), then $\mathcal{E}(X)$ is non negative (resp. positive). Moreover, if X is a local martingale, then $\mathcal{E}(X)$ is a local martingale.

A.1.2 A p-integrability property with $p > 2$, of local exponential martingales driven by a Brownian motion and a Poisson random measure

We now show the following property, which is used in Section 5.3, to solve a robust optimization problem, where some p-integrability conditions, with $p > 2$, are required.

Proposition A.1 *Let (β_t) and $(\gamma_t(.))$ be predictable $\mathbb{R}$-valued processes and let M be the local martingale defined by $M_t := \int_0^t \beta_s dW_s + \int_0^t \int_{\mathbb{R}^*} \gamma_s(u)\tilde{N}(ds,du)$. Suppose β is bounded and that, $dt \otimes P$-a.s.,*

$$|\gamma_t(u)| \leq \psi(u) \ \ d\nu(u) - \ \ a.s. \tag{A.70}$$

where $\psi : \mathbf{R}^* \to \mathbf{R}^+$ is a Borelian function.

a) Suppose that $\mathcal{E}(M) \geq 0$.
For each integer $p \geq 2$, the following assertion holds:
if $\psi \in L^p_\nu$, then $E[\mathcal{E}(M)^p_T] < +\infty$.

b) Otherwise, the above assertion holds for each even number $p \geq 2$.

Moreover, if $\psi \in L^p_\nu$ for all $p \geq 1$, then $\mathcal{E}(M)_T$ is p-integrable for all $p \geq 1$.

Proof. Let us show by induction Property a). Suppose that $\mathcal{E}(M) \geq 0$ a.s. We have already shown that the property holds for $p = 2$. To avoid heavy notation and for the clarty of the proof, we first show the property for $p = 3$ before proceeding with the induction. Suppose that $\psi \in L^3_\nu$. We have to prove that $E[\mathcal{E}(M)^3_T] < +\infty$.
By equality (3.5), it is sufficient to show that $E[\mathcal{E}(N)_T \mathcal{E}(M)_T] < +\infty$, with

$$N_t = 2M_t + \int_0^t \int_{\mathbb{R}^*} \gamma_s^2(u)\tilde{N}(ds,du) = \int_0^t 2\beta_s dW_s + \int_0^t \int_{\mathbb{R}^*} \left(2\gamma_s(u) + \gamma_s^2(u)\right)\tilde{N}(ds,du).$$

Using more concise notation, N can be written $N = 2\beta.W + (2\gamma + \gamma^2).\tilde{N}$.
By the product formula, $\mathcal{E}(N)\mathcal{E}(M) = \mathcal{E}(N + M + [N,M])$. Now, by classical properties of $[.,.]$, we get

$$[N,M] = [2\beta.W + (2\gamma + \gamma^2).\tilde{N}, \beta.W + \gamma.\tilde{N}] = 2\beta^2.dt + (2\gamma^2 + \gamma^3).N$$

Let N_3 be the local martingale given by $N_3 = N + M + (2\gamma^2 + \gamma^3).\tilde{N}$. We thus have

$$\mathcal{E}(N)\mathcal{E}(M) = \mathcal{E}(N_3)\exp\{2\int_0^\cdot \beta_s^2 ds + \int_0^\cdot \int_{\mathbf{R}^*} (2\gamma_s^2 + \gamma_s^3)\nu(du)\, ds\}, \tag{A.71}$$

We have seen in the proof of Proposition 3.2 that $\mathcal{E}(N) \geq 0$ and, by assumption, we have $\mathcal{E}(M) \geq 0$. This together with the above equality yields that $\mathcal{E}(N_3) \geq 0$. Since $\mathcal{E}(N_3)$ is also a local martingale, it follows that it is a supermartingale. Using the assumption $\psi \in L^3_\nu$, we get $E[\mathcal{E}(N)_T\mathcal{E}(M)_T] \leq E[\mathcal{E}(N_3)_T]\, K \leq K$, where K is a positive constant, which yields that $E[\mathcal{E}(M)^3_T] < +\infty$.

Let us show by induction that for all $p \geq 2$, $\mathcal{E}(M)^p = \mathcal{E}(N_p)\exp\{\int_0^\cdot a_p(s)ds\}$, where a_p is a bounded adapted process and N_p is a local martingale given by $N_p = \alpha_p.W + Q_p(\gamma).\tilde{N}$, such that $\mathcal{E}(N_p) \geq 0$. Here, α_p is a bounded predictable process and Q_p a polynomial with degree p. Suppose we have shown this property for some $p \geq 2$ and let us show that it still holds at rank $p+1$. Using the induction hypothesis, we get

$$\mathcal{E}(M)^{p+1} = \mathcal{E}(M)^p\mathcal{E}(M) = \mathcal{E}(N_p)\mathcal{E}(M)\exp\{\int_0^\cdot a_p(s)ds\}.$$

Now, $\mathcal{E}(N_p)\mathcal{E}(M) = \mathcal{E}(N_p + M + [N_p, M])$ and

$$[N_p, M] = [\alpha_p.W + Q_p(\gamma).\tilde{N}, \beta.W + \gamma.\tilde{N}] = \alpha_p\beta.dt + Q_p(\gamma)\gamma.N.$$

Let N_{p+1} be the local martingale given by $N_{p+1} = N_p + M + Q_p(\gamma)\gamma.\tilde{N}$. Then, using these equalities, we derive the desired property at rank $p+1$.

Using this result and similar arguments as above, one can derive Property a).

Let us show Property b). First, we have already shown that the property holds for $p = 2$. Let us now show the property for $p = 2q$, with $q \geq 2$. Suppose $\psi \in L_\nu^{2q}$ and let us prove that $E[\mathcal{E}(M)_T^{2q}] < +\infty$. By equality (3.5), it is sufficient to show that $E[\mathcal{E}(N)_T^q] < +\infty$. Now, $\mathcal{E}(N) \geq 0$ a.s. Applying the first assertion with M replaced by N and using the fact that $\psi \in L_\nu^{2q}$, we derive the desired result.

The last assertion of Proposition A.1 follows from Property b).

$\square$

Example: Suppose that the intensity measure ν of the Poisson random measure satisfies

$$\int_{\mathbf{R}^*}(1 \wedge u^2)\nu(du) < +\infty$$

Let (β_t) and $(\gamma_t(.))$ be predictable $\mathbb{R}$-valued processes and let M be the local martingale defined by (3.2). Suppose β is bounded and that, $dt \otimes P$-a.s.,

$$|\gamma_t(u)| \leq K(1 \wedge |u|) \ \ d\nu(u) - \text{ a.s. } ,$$

where K is a positive constant. Then, for all $p \geq 2$, $|\gamma_t(u)|^p \leq K^p(1 \wedge |u|^p) \leq K^q(1 \wedge |u|^2)$. Hence, for all $p \geq 1$, $\mathcal{E}(M)_T$ is p-integrable.

We point out that these properties of exponential local martingales can also be useful in the Option Pricing Theory, especially in the study of local martingale measures.

A.2 Some complementary results on BSDEs with jumps

A.2.1 BSDEs with jumps in L^p, $p \geq 2$

Proposition A.5 *Let $p \geq 2$ and let $T > 0$. For each Lipschitz driver f, and each terminal condition $\xi \in L^p(\mathcal{F}_T)$, there exists a unique solution (X, π, l) $\in S^{p,T} \times \mathbb{H}^{p,T} \times \mathbb{H}_\nu^{p,T}$ of the BSDE with jumps (2.1).*

Remark A.10 *The above property still holds in the case when there is an $\mathbb{F}$-martingale representation theorem with respect to W and $\tilde{N}$, even if $\mathbb{F}$ is not generated by W and $\tilde{N}$.*

Proof. Let us first consider the case when the driver f does not depend on x, π, ℓ.
Then, X is given by the right-continuous version of $X_t = E[\xi + \int_t^T f(s)ds \mid \mathcal{F}_t]$. Also, since $p \geq 2$, by the martingale representation theorem of Tang and Li [26] for locally square integrable martingales, (π, l) corresponds to the unique pair of predictable processes satisfying

$$E[\xi + \int_0^T f(s)ds \mid \mathcal{F}_t] = X_0 + \int_0^t \pi_s dW_s + \int_0^t \int_{\mathbf{R}^*} l_s(u)\tilde{N}(ds, du) \quad \text{a.s.} \tag{A.72}$$

We have $|X_t| \leq E[|\xi| + \int_0^T |f(s)|ds \mid \mathcal{F}_t]$. Hence, using martingale inequalities, we get

$$\|X\|_{S^{p,T}}^p \leq C_p E\left[\left(|\xi| + \int_0^T |f(s)|ds\right)^p\right],$$

where C_p is a constant which does not depend on T. We derive that

$$\|X\|_{S^{p,T}}^p \leq C_p E\left[|\xi|^p + T^{\frac{p}{2}}(\int_0^T f(s)^2 ds)^{\frac{p}{2}}\right] = C_p\left(E[|\xi|^p] + T^{\frac{p}{2}}\|f\|_{\mathbb{H}^{p,T}}^p\right), \tag{A.73}$$

for another constant still denoted by C_p. Also, by Burkohlder-Davis-Gundy inequalities, since $p > 1$, we have

$$E\left[\left(\int_0^T \pi_s^2 ds + \int_0^T \|l_s\|_\nu^2 ds\right)^{\frac{p}{2}}\right] \leq C_p E\left[\left(|\int_0^T \pi_s dW_s + \int_0^T \int_{\mathbf{R}^*} l_s(u)\tilde{N}(ds, du)|\right)^p\right].$$

Equality (A.72) together with the above estimates lead to

$$\|\pi\|_{\mathbb{H}^{p,T}}^p + \| l\|_{\mathbb{H}_\nu^{p,T}}^p \leq C_p\left(E(|\xi|^p) + T^{\frac{p}{2}}\|f\|_{\mathbb{H}^{p,T}}^p\right). \tag{A.74}$$

Let us consider the case of a general driver $f(t, x, \pi, \ell)$. Denote by $\mathcal{H}^{p,T}$ the Banach space $S^{p,T} \times \mathbb{H}^{p,T} \times \mathbb{H}_\nu^{p,T}$ equipped with the norm $\|(X, \pi, l)\|_{p,T} :=$

$\|X\|_{S^{p,T}} + \|\pi\|_{\mathbb{H}^{p,T}} + \|l\|_{\mathbb{H}_\nu^{p,T}}$.
Let us define the map Ψ from $\mathcal{H}^{p,T}$ into itself as follows. Given $(U, V, k) \in \mathcal{H}^{p,T}$, let $(X, \pi, l) = \Phi(U, V, k)$ be the unique element of $\mathcal{H}^{p,T}$ solution of the BSDE associated with driver $f(s) = f(s, U_s, V_s, k_s)$. Let us prove that Ψ is a contraction for T enough small.
Let (U', V', k') be another element of $\mathcal{H}^{p,T}$ and define $(X', \pi', l') = \Psi(U', V', k')$. Set $\bar{U} = U - U'$, $\bar{V} = V - V'$, $\bar{k} = k - k'$, $\bar{X} = X - X'$, $\bar{\pi} = \pi - \pi'$ $\bar{l} = l - l'$. The process $(\bar{X}, \bar{\pi}, \bar{l})$ is the solution of the BSDE associated with terminal condition 0 and the driver $\Delta f. = f(\cdot, U, V, k) - f(\cdot, U', V', k')$. By inequalities (A.73) and (A.74), we derive that

$$\|\bar{X}\|_{S^{p,T}}^p + \|\bar{\pi}\|_{\mathbb{H}^{p,T}}^p + \|\ \bar{l}\|_{\mathbb{H}_\nu^{p,T}}^p \leq C_p T^{\frac{p}{2}} \|f(\cdot, U, V, k) - f(\cdot, U', V', k')\|_{\mathbb{H}^{p,T}}^p.$$

Using the Lipschitz property of f, we get, for another constant still denoted by C_p,

$$\|\bar{X}\|_{S^{p,T}} + \|\bar{\pi}\|_{\mathbb{H}^{p,T}} + \|\ \bar{l}\|_{\mathbb{H}_\nu^{p,T}} \leq C_p \sqrt{T} \left(\sqrt{T} \|\bar{U}\|_{S^{p,T}} + \|\bar{V}\|_{\mathbb{H}^{p,T}} + \|\ \bar{k}\|_{\mathbb{H}_\nu^{p,T}} \right).$$

Choosing T such that $C_p\sqrt{T} < 1$ and $C_p T < 1$, the map Ψ is a contraction on the Banach space $\mathcal{H}^{p,T}$ and hence admits a fixed point which corresponds to the solution of BSDE (2.1) in $\mathcal{H}^{p,T}$.
The general case is obtained by subdivising the interval $[0, T]$ into a finite number of small intervals. □

A.2.2 Estimates and continuity result

Proposition A.6 (Estimates) *Let $T > 0$ and let ξ^1, $\xi^2 \in L^2(\mathcal{F}_T)$. Let f^1 be a Lipschitz driver with Lipschitz constant C and let f^2 be a driver. For $i = 1, 2$, let (X^i, π^i, l^i) be a solution of the BSDE (2.1) associated to terminal time T, driver f^i and terminal condition ξ^i. For s in $[0, T]$, denote $\bar{X}_s := X_s^1 - X_s^2$, $\bar{\pi}_s := \pi_s^1 - \pi_s^2$, $\bar{l}_s := l_s^1 - l_s^2$, and $\bar{f}(s) := f^1(s, X_s^2, \pi_s^2, l_s^2) - f^2(s, X_s^2, \pi_s^2, l_s^2)$ and $\bar{\xi} := \xi^1 - \xi^2$.*

Let $\eta, \beta > 0$ be such that $\beta \geq \frac{3}{\eta} + 2C$. If $\eta \leq \frac{1}{C^2}$, then, for each $t \in [0, T]$, we have

$$e^{\beta t} \bar{X}_t^2 \leq E[e^{\beta T} \bar{\xi}^{\,2} \mid \mathcal{F}_t] + \eta\, E[\int_t^T e^{\beta s} \bar{f}(s)^2 ds \mid \mathcal{F}_t] \quad a.s.; \tag{A.75}$$

$$\|\bar{X}\|_\beta^2 \leq T[e^{\beta T} E[\bar{\xi}^{\,2}] + \eta \|\bar{f}\|_\beta^2]. \tag{A.76}$$

Also, if $\eta < \frac{1}{C^2}$, we then have

$$\|\bar{\pi}\|_\beta^2 + \|\bar{l}\|_{\nu,\beta}^2 \leq \frac{1}{1 - \eta C^2} [e^{\beta T} E[\bar{\xi}^{\,2}] + \eta \|\bar{f}\|_\beta^2]. \tag{A.77}$$

Proof.

From Itô's formula applied to the semimartingale $e^{\beta s}\bar{X}_s^2$ between t and T, it follows that

$$\begin{aligned}
&e^{\beta t}\bar{X}_t^2 + \beta \int_t^T e^{\beta s}\bar{X}_s^2 ds + \int_t^T e^{\beta s}\bar{\pi}_s^2 ds + \int_t^T e^{\beta s}\|\bar{l}_s\|_\nu^2 ds \\
&\quad = e^{\beta T}\bar{X}_T^2 + 2\int_t^T e^{\beta s}\bar{X}_s(f^1(s, X_s^1, \pi_s^1, l_s^1) - f^2(s, X_s^2, \pi_s^2, l_s^2))ds \\
&\quad - 2\int_t^T e^{\beta s}\bar{X}_s\bar{\pi}_s dW_s - 2\int_t^T e^{\beta s}\int_{\mathbb{R}}^* \bar{X}_{s^-}\bar{l}_s(u)d\tilde{N}(du, dt).
\end{aligned}$$

Now, by martingale inequalities, one can show that X_1 and X_2 belong to S^2, which provide that the local martingales of the right hand side of the above equality are martingales. Moreover,

$$\begin{aligned}
|f^1(s, X_s^1, \pi_s^1, l_s^1) - f^2(s, X_s^2, \pi_s^2, l_s^2)| &\leq |f^1(s, X_s^1, \pi_s^1, l_s^1) - f^1(s, X_s^2, \pi_s^2, l_s^2)| + |\bar{f}_s| \\
&\leq C|\bar{X}_s| + (C|\bar{\pi}_s| + C\|\bar{l}_s\|_\nu + |\bar{f}_s|).
\end{aligned}$$

Now, for all real numbers x, π, l, f and $\epsilon > 0$
$2x(C\pi + Cl + f) \leq \frac{x^2}{\epsilon^2} + \epsilon^2(C\pi + Cl + f)^2 \leq \frac{x^2}{\epsilon^2} + 3\epsilon^2(C^2\pi^2 + C^2l^2 + f^2)$.
Hence, we get

$$\begin{aligned}
&e^{\beta t}\bar{X}_t^2 + E\left[\beta\int_t^T e^{\beta s}\bar{X}_s^2 ds + \int_t^T e^{\beta s}(\bar{\pi}_s^2 + \|\bar{l}_s\|_\nu^2)ds \mid \mathcal{F}_t\right] \\
&\quad \leq \mathbb{E}\left[e^{\beta T}(\xi_1 - \xi_2)^2 + (2C + \frac{1}{\epsilon^2})\int_t^T e^{\beta s}\bar{X}_s^2 ds + 3C^2\epsilon^2\int_t^T e^{\beta s}(\bar{\pi}_s^2 + \|\bar{l}_s\|_\nu^2)ds \mid \mathcal{F}_t\right] \\
&\quad + 3\epsilon^2\mathbb{E}\left[\int_t^T e^{\beta s}\bar{f}_s^2 ds \mid \mathcal{F}_t\right]. \qquad \text{(A.78)}
\end{aligned}$$

Let us make the change of variable $\eta = 3\epsilon^2$. Then, for each $\beta, \eta > 0$ chosen as in the theorem, these inequalities lead to (A.75).

We obtain (A.76) by integrating (A.75). Then (A.77) follows from inequality (A.78). □

Remark A.11 *By classical results on the norms of semimartingales, one similarly shows that $\|\bar{X}\|_{S^2} \leq K\left(\mathbb{E}[\bar{\xi}^2] + \|\bar{f}\|_{\mathbb{H}^2}\right)$, where K is a positive constant only depending on T.*

We denote by $(X(\xi,T),\pi(\xi,T),l(\xi,T))$ the solution of the BSDE associated with f, terminal time $T>0$, and terminal condition $\xi\in L^2(\mathcal{F}_T)$.

Let S be a stopping time smaller than T and let $\xi\in\mathcal{F}_S$. Let $(X(\xi,S),\pi(\xi,S),l(\xi,S))$ (denoted here by (X,π,l)) be the solution associated with f, terminal time S and terminal condition ξ. By conventional notation, we extend the solution on the whole interval $[0,T]$ by setting $X_t=\xi,\pi_t=0,l_t=0$ for $t\geq S$. So, $((X_t,\pi_t,l_t);t\leq T)$ is the unique solution of the BSDE with driver $f(t,x,\pi,l)\mathbf{1}_{\{t\leq S\}}$ and terminal conditions $(T,\,\xi)$.

By the uniqueness result (as in the Brownian case (see [10])), we derive the *flow property* for BSDEs with jumps. More precisely,

$$X_t(\xi,T)=X_t(X_S(\xi,T),S),\quad t\in[0,S],\quad \text{a.s.} \tag{A.79}$$

and the same property holds for $\pi(\xi,T)$ and $l(\xi,T)$. This property ensures the consistency property of the associated risk measure ρ^f.

Now, the above estimates allow us to show the continuity property of $X(\xi,T)$ with respect to (ξ,T), which yields the continuity property of the associated dynamic risk measure ρ^f with respect to (ξ,T).

Proposition A.7 (A continuity result.) *Let $T>0$. Let $\{\theta^\alpha,\alpha\in\mathbf{R}\}$ be a family of stopping times in $\mathcal{T}_0$, converging a.s. to a stopping time $\theta\in\mathcal{T}_0$ as α tends to α_0. Let $(\xi^\alpha,\alpha\in\mathbf{R})$ be a family of random variables such that $\mathbb{E}[\operatorname{ess\,sup}_\alpha(\xi^\alpha)^2]<+\infty$, and for each α, ξ^α is $\mathcal{F}_{\theta^\alpha}$-measurable. Suppose also that ξ^α converges a.s. to an $\mathcal{F}_\theta$-measurable random variable ξ as α tends to α_0. Let f be a given standard driver. Let $X^\alpha:=X(\xi^\alpha,\theta^\alpha)$; $\pi^\alpha:=\pi(\xi^\alpha,\theta^\alpha)$; $l^\alpha:=l(\xi^\alpha,\theta^\alpha)$; and $X:=X(\xi,\theta)$; $\pi:=\pi(\xi,\theta)$; $l:=l(\xi,\theta)$. Then, for each $S\in\mathcal{T}_0$, the random variable X^α_S converges to X_S a.s and the process X^α converges to X in $S^{2,T}$.*

Proof. By the convention given above, (X,π,l) is the solution associated to BSDE with terminal time T, terminal condition ξ and driver $f(t,x,\pi,l)\mathbf{1}_{t\leq\theta}$. Also, $(X^\alpha,\pi^\alpha,l^\alpha)$ is the solution associated to BSDE with terminal time T, terminal condition ξ^α and driver $f(t,x,\pi,l)\mathbf{1}_{t\leq\theta^\alpha}$. Applying the estimate (A.75), we get, for each stopping time S,

$$e^{\beta S}(X_S-X^\alpha_S)^2\leq E[e^{\beta T}(\xi-\xi^\alpha)^2+\eta\int_{(\theta^\alpha\wedge\theta)\vee S}^{(\theta^\alpha\vee\theta)\wedge S}e^{\beta s}f(s,X_s,\pi_s,l_s)^2ds\mid\mathcal{F}_S]$$

with β and η as in Theorem A.6. By the assumptions and the Lebesgue theorem, we conclude that X^α_S converges to X_S a.s. Moreover, by Remark A.11, the process X^α converges to X in $S^{2,T}$. $\square$

We stress that the two above properties, in particular estimate (A.75), are also very useful to study optimal stopping problems for dynamic risk measures (see [24]). Note that, even in the Brownian case, estimate (A.75) had not be written explicitly in the literature.

A.2.3 Proof of the convexity property (5.40)

Let $\lambda \in (0,1)$ and let ξ^1 and $\xi^2 \in L^2(\mathcal{F}_T)$.
We want to prove that $\lambda X(\xi^1, T) + (1-\lambda) X(\xi^2, T) \leq X(\lambda \xi^1 + (1-\lambda)\xi^2, T)$.
Let (X^1, π^1, l^1) (resp. (X^2, π^2, l^2)) be the solution of the BSDE associated with driver f and terminal condition ξ^1 (resp. ξ^2). By taking the linear combination of these two BSDEs with coefficients λ and $1-\lambda$, we derive that the process $(\tilde{X}, \tilde{\pi}, \tilde{\lambda})$ defined by

$$\tilde{X} = \lambda X^1 + (1-\lambda) X^2 \; ; \; \tilde{\pi} = \lambda \pi^1 + (1-\lambda)\pi^2 \; ; \; \tilde{l} = \lambda l^1 + (1-\lambda) l^2.$$

is the solution of the BSDE associated with terminal condition $\lambda\xi^1 + (1-\lambda)\xi^2$ and the driver given by the predictable process $\tilde{f}_t = \lambda f(t, X_t^1, \pi_t^1, l_t^1) + (1-\lambda) f(t, X_t^2, \pi_t^2, l_t^2)$.
Let now $(X, \pi,)$ be the solution of the BSDE associated with driver f and terminal condition $\lambda\xi^1 + (1-\lambda)\xi^2$. By concavity of f, we have

$$\tilde{f}_t = \lambda f(t, X_t^1, \pi_t^1, l_t^1) + (1-\lambda) f(t, X_t^2, \pi_t^2, l_t^2) \leq f(t, \tilde{X}_t, \tilde{\pi}, \tilde{l}_t), \; t \in [0,T], \; dP \otimes dt - a.s.$$

Using the comparison theorem (4.2), we conclude $\lambda X^1 + (1-\lambda) X^2 = \tilde{X} \leq X$ a.s.
□

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